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The collective effect of rework, expedited-rate, external source, and machine failures on manufacturing runtime planning

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A B S T R A C T

Production managers face the growing trend of rapid-response orders and inevitable production defects and failures; they must carefully measure these factors' effects to minimize operating expenditures and operational disruption. Inspired by assisting producers decide the optimal runtime policy under these real situations, this work investigates the collective impact of rework, expedited-rate, external source, and machine failures on such a specific fabrication system. A partial outsourcing and expedited manufacturing rate are considered in the studied system to reduce the batch fabricating time. Additionally, defects rework and repair failure machines are implemented to retain the quality and avoid production disruption. Our research scheme consists of (1) developing a model for the mentioned manufacturing characteristics; and (2) analytical and optimization techniques for deciding the best batch runtime decision by minimizing the system's overall expenses. Lastly, we provide numerical examples to demonstrate the model's applicability and disclose important, in-depth characteristics that facilitate managerial decision-making.

1. Introduction

Facing inevitable production defects and machine failures and the rising trend of rapid-response orders, current manufacturers must carefully measure the above factors' impact to minimize the potential operational disruption and the total operating expenditures. This work develops a model to cautiously explore the collective effect of rework, machine failures, and implementing expedited-rate and external sources on such a specific fabrication system to facilitate managerial decision-making. Instantly reworking random defects and correcting machine failures can maintain anticipated product quality and prevent unwanted fabrication interruption. Jabal Ameli et al. (2008) studied a cell formation problem featuring unreliable machine and substitute process routings. The researchers proposed a multi-objective cost-minimization and reliability- maximization model and applied integer linear programming techniques to resolve it. They mainly explored the cost and time-based effects of the unreliable machine and applied an É-constraint methodology for optimizing their multi-objective programming. Finally, the researchers demonstrated the capability of their model via numerical examples and evaluated various influences of reliability considerations. Ullah and Kang (2014) examined an inventory model considering imperfect production with inspection, rejects, rework, and work in process. The researchers built a model for

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determining the optimum batch size that kept the average system cost minimum. The impact of the above-mentioned practical system features are assessed via numerical examples, and the research results are compared and discussed with existing models. Öztürk, H. (2021) incorporated product screening time, rework, and the facility failures into an imperfect fabrication-inventory integrated system, wherein the manufacturing runtime is a decision variable. The scenario considered a continuous screening process and no allowable stock-out situation; the author developed a math model and showed the concavity of the profit function. The study obtained an optimal system policy and examined a few relevant exceptional cases by applying an analytical methodology. Finally, the researcher draws insights into the study with a numerical illustration with sensitivity analysis. Extra works (Kumar et al., 2004; Maggio et al., 2009; Rostami et al., 2018; Karim and Nakade, 2021; Yamada et al., 2021) discovered the effect of different unreliable machine situations and rework policies on operating control and management for manufacturing systems.

To cope with the rising trend of rapid-response orders, the present study considers partial outsourcing and an expedited rate in our fabrication model, aiming to shorten batch fabricating time. Outsourcing and expedited-rate relating surveys include: Mendelson and Parlaktürk (2008) explored the market competition focusing on product price and variety issues. The researchers considered two different producers. The traditional one has a narrow set of products with on-hand inventories and an aggressive (customizing) firm that will take the client's order of any configured product with no no-hand stocks available. They further assumed that in the short run, the aggressive producer has a limited capacity. The equilibrium for a duopoly competition of these different types of producers was derived. The characteristics and monopoly were analyzed and compared. Through their further investigation, the researchers disclosed the managerial insights of the degree of customization, market size, stock holding cost, capacity expanding, product variety, and profit. Ayed et al. (2011) derived the optimal production policy for an integrated production-maintenance system with stochastic demand, service-level constraint, variable fabricating rate, and subcontracting. They assumed the in-house capacity was limited. Their study incorporated the necessary subcontracting and possibly increased the fabricating rate to meet the uncertain demands and service levels. By considering the above factors plus the degrading machine issue, the researchers aimed to decide on an optimal hybrid production plan that meets the demand and minimizes the relevant costs. Neidigh and Harrison (2017) derived the optimal batch size for a multiproduct multi-machine system with increasing (nonlinear) fabricating rates to create production efficiency owing to learning effects. Their study was particularly suitable for the large batch sizes, but it was ill-suited to the just-in-time application where lot size is small. The researchers built a model to balance the influence of several competing factors and aim to decide the best lot sizes to meet demands and minimize the fabrication-inventory cost. Finally, they extended the original model to deal with the multiproduct, multi-machine manufacturing system and demonstrated the efficiency in obtaining the results with actual cases. Hazrati et al. (2021) explored an economic order problem by developing a hybrid decision model to minimize operating costs and maximize the number of outsourced products with different weight values according to the fuzzy analytic method. Their model considered simultaneous orders of multiproduct from multiple suppliers in batch and with discount. The researchers used a non-dominated sorting algorithm from MATLAB to solve this multi-objective model. They validated the research results with the meta-heuristic solution and found that it fell within the 1% range of the optimal solution. The study also provided managerial insights on the problem concerning demand, discount, and overall operating cost. Extra works (Grossman and Helpman, 2002; Pan and Yang, 2008; Shy and Stenbacka, 2012; Chiu et al., 2019; Dey et al., 2019; Ramasubbu et al., 2019; Chiu et al., 2020; Kershaw et al., 2021; Chiu et al., 2021) explored the impact of different subcontracting strategies and variable fabricating rates on diverse manufacturing systems and their production controlling, planning, and management. Few works have studied the collective effect of rework, expedited rate, external source, and machine failures on manufacturing runtime planning; we try to fill this gap.

2. Assumption, description, and modeling of the problem

This work explores the collective effect of rework, expedited-rate, external source, and machine failures on manufacturing runtime planning. The relevant definition of symbols is provided below.

π = outsourcing proportion of a batch (where $0 < \pi < 1$),
C_π = unit outsourcing cost,
K_π = outsourcing setup cost,

C = standard unit cost,
K = standard setup cost,
β_2 = the connecting variable between C and C_π,
β_1 = the connecting variable between K and K_π,
P_{1A} = expedited rate per year,
P_{2A} = annual expedited reworking rate,
P_2 = standard annual reworking rate,
C_A = unit cost when expedited-rate is implemented,
K_A = setup cost when expedited-rate is implemented,
P_1 = standard producing rate (i.e., without implementing expedited rate),
P_2 = standard rework rate,
C_{RA} = unit rework cost when P_{2A} is implemented,
C_R = standard reworking cost,
α_1 = the connecting variable between P_1 and P_{1A}, and between P_2 and P_{2A},
α_2 = the connecting variable between K and K_A,
α_3 = the connecting variable between C_A and C, and between C_{RA} and C_R,
β = mean Poisson-distributed failures per year,
t = mean time to breakdown,
t_r = needed/allowed time to fix a failure,
M = cost for fixing a failure,
λ = annual demand,
t_{1Z} = production runtime/uptime – the decision variable,
Q = production lot-size,
t'_{2Z} = rework time in the failure happening case,
t'_{3Z} = finished-items depleting time in the failure happening case,
T'_Z = cycle length in the failure happening case,
x = Uniform-distributed annual defective rate,
d_{1A} = annual production rate of nonconforming items, where $d_{1A} = P_{1A}\,x$,
h = unit holding cost,
h_3 = unit holding cost of safety stock,
h_1 = reworked item's unit holding cost,
C_1 = unit cost of safety stock,
C_T = unit delivery cost,
g = t_r, needed/allowed time to fix a failure,
H = inventory level when the outsourced goods are received,
H_0 = inventory level when a failure happens,
H_1 = inventory level when production uptime finishes,
H_2 = inventory level when rework time finishes,
T_Z = cycle length when no breakdown happening,
t_{2Z} = rework time when no breakdown happening,
t_{3Z} = finished-items depleting time in the no failure happening case,
T = cycle length for a system without expedited-rate, external source, nor failures,
t_1 = uptime for a system without expedited-rate, external source, nor failures,
t_2 = rework time for a system without expedited-rate, external source, nor failures,
t_3 = finished-items depleting time for a system without expedited-rate, external source, nor failures,
d_1 = annual production rate of nonconforming items for a system without expedited-rate, external source, nor failures,
$I(t)$ = inventory level at time t,
$I_F(t)$ = inventory level of safety stock at time t,
$I_d(t)$ = defective inventory level at time t,
$TC(t_{1Z})_1$ = total cost per cycle in the breakdown occurring case,
$TC(t_{1Z})_2$ = total cost per cycle in no breakdown occurring case,
$E[TC(t_{1Z})_1]$ = the expected total cost per cycle in case one of this study,
$E[TC(t_{1Z})_2]$ = the expected total cost per cycle in case two,
$\boldsymbol{T_Z}$ = cycle length,
$E[T_Z]$ = the expected cycle length in case two of this study,
$E[T'_Z]$ = the expected cycle length in case on,

$E[TCU(t_{1Z})]$ = the expected system cost.

Consider the proposed batch production system has a lot-size Q, and it needs to meet the product demand of λ per year. An external source helps supply a πQ portion of the lot to reduce the batch cycle length. To further shorten cycle length, the in-house process uses an expedited rate P_{1A} to manufacture the other $(1-\pi)Q$ of the lot. The following relationships accompanying the expedited-rate strategy versus standard production:

$$P_{1A} = (1+\alpha_1)P_1 \tag{1}$$

$$C_A = (1+\alpha_3)C \tag{2}$$

$$K_A = (1+\alpha_2)K \tag{3}$$

Eq. (4) and Eq. (5) exhibit the relationships accompanying the outsourcing strategy versus in-house production:

$$C_\pi = (1+\beta_2)C \tag{4}$$

$$K_\pi = (1+\beta_1)K \tag{5}$$

In each cycle, the reworking of x proportion of defective products randomly produced by the in-house process ensures the desired product quality. On the other hand, the external source promises their products' quality. The scheduled receipt time of outsourcing products is at the beginning time of the in-house stock depleting time. This study does not permit stock-out situations; so, $(P_{1A} - d_{1A} - \lambda) > 0$. Eq. (6) and Eq. (7) show the relationships of parameters accompanying the expedited reworking rate versus the standard one:

$$P_{2A} = (1+\alpha_1)P_2 \tag{6}$$

$$C_{RA} = (1+\alpha_3)C_R \tag{7}$$

Furthermore, the production facility is subject to a Poisson distribution breakdown-rate with a mean of β failures per year. The time to a failure occurrence t adheres to the Exponential-distributed rate (i.e., $f(t) = \beta e^{-\beta t}$ as its density function). This study adopts an abort/resume (A/R) stock control policy when a failure happens. The fabrication of interrupted (unfinished) lot immediately resumes when the failure is corrected. This study assumes a fixed failure-repair time t_r; if actual repair time exceeds t_r, we use a rental/spare machine to avoid unwanted delay in the production. To explicitly explore the randomness of equipment failures, this study considers the following separate cases:

2.1. Case one: A random failure happens during uptime

In case one, the time to a failure incidence $t < t_{1Z}$. Fig. 1 exhibits the case one's stock level (in blue lines).

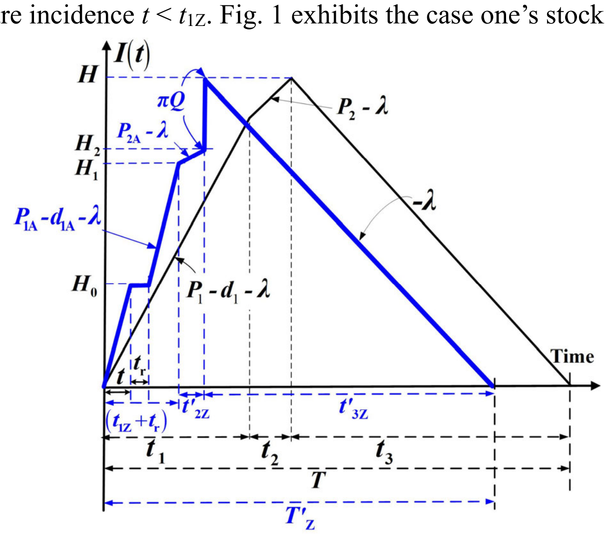

Fig. 1. The case one's stock level (in blue lines) compared to the same problem but without the uptime-reduction strategies (in black lines)

When a failure occurs, the stock status arrives at H_0. Its position stays the same during t_r. After the failure is corrected, the inventory level grows up again, and it reaches H_1 when t_{1Z} ends. Then, the rework process brings the inventory level to H_2 when t'_{2Z} ends. The external source supplies the outsourced items to bring up the inventory level to H at the beginning of depleting time t'_{3Z}. Once the stock level drops to zero at the end of t'_{3Z}, the next replenishing cycle begins (refer to Fig. 1). Fig. 2 exhibits the safety inventory level in case one. The proposed model utilizes the safety stocks to satisfy the product demand during t_r. Fig. 3 illustrates the status of defective products. The total defective products in a cycle are as follows:

$$d_{1A}t_{1Z} = xQ(1-\pi) = xP_{1A}t_{1Z} \tag{8}$$

Also, the following relationships are observed from Fig. 1 to Fig. 3:

$$H_0 = (P_{1A} - d_{1A} - \lambda)t \tag{9}$$

$$t_{1Z} = \frac{H_1}{P_{1A} - d_{1A} - \lambda} = \frac{(1-\pi)Q}{P_{1A}} \tag{10}$$

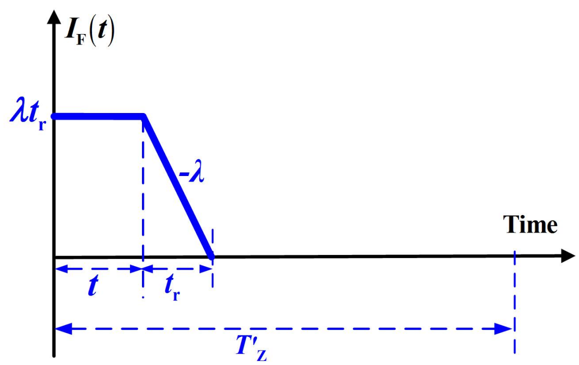

Fig. 2. The safety inventory level in case one

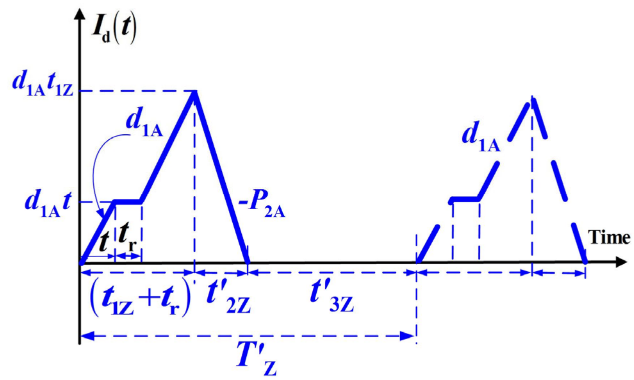

Fig. 3. The status of defective products in case one during T'_Z

$$H_1 = (P_{1A} - d_{1A} - \lambda)t_{1Z} \tag{11}$$

$$t'_{2Z} = \frac{x\left[(1-\pi)Q\right]}{P_{2A}} \tag{12}$$

$$H_2 = H_1 + (P_{2A} - \lambda)t'_{2Z} \tag{13}$$

$$H = H_2 + \pi Q \tag{14}$$

$$t'_{3Z} = \frac{H}{\lambda} \tag{15}$$

$$T'_Z = t_{1Z} + t_r + t'_{2Z} + t'_{3Z} \tag{16}$$

$TC(t_{1Z})_1$, the total cost per cycle comprises the following: the fixed and variable in-house manufacturing and outsourcing cost, safety stock relevant cost, failure correction cost, reworking cost, and holding costs (including the reworked items, perfect and defective products) during T'_Z (see Eq. (17)).

$$\begin{aligned} TC(t_{1Z})_1 &= K_A + (1-\pi)QC_A + K_\pi + (\pi Q)C_\pi + C_1(\lambda t_r) + M + x(1-\pi)QC_{RA} + C_T(\lambda t_r) \\ &+ (\lambda t_r)\left(t + \frac{t_r}{2}\right)h_3 + \frac{P_{2A}t'_{2Z}}{2}(t'_{2Z})h_1 \\ &+ h\left[\frac{H_1 + d_{1A}t_{1Z}}{2}(t_{1Z}) + \frac{H_1 + H_2}{2}(t'_{2Z}) + \frac{H}{2}(t'_{3Z}) + (H_0 t_r) + (d_{1A}t)t_r\right] \end{aligned} \tag{17}$$

Substitute Eqs. (1) to (8) in Eq. (17), $TC(t_{1Z})_1$ is as follows:

$$\begin{aligned} TC\left(t_{1Z}\right)_1 &= \left(1+\alpha_2\right)K + \left(1-\pi\right)Q\left(1+\alpha_3\right)C + \left(1+\beta_1\right)K + C\left(\pi Q\right)\left(1+\beta_2\right) + M + x\left(1-\pi\right)Q\left(1+\alpha_3\right)C_{\mathrm{R}} \\ &+ \left(\lambda t_r\right)C_1 + \left(\lambda t_r\right)C_T + \left(\lambda t_r\right)\left(t+\frac{t_r}{2}\right)h_3 + \frac{\left[\left(1+\alpha_1\right)P_2\right]t'_{2Z}}{2}\left(t'_{2Z}\right)h_1 \\ &+ h\left[\frac{H_1 + x\left(1+\alpha_1\right)P_1 t_{1Z}}{2}\left(t_{1Z}\right) + \frac{H_1+H_2}{2}\left(t'_{2Z}\right) + \frac{H}{2}\left(t'_{3Z}\right) + \left(H_0 t_r\right) + x\left(1+\alpha_1\right)P_1\left(t\right)t_r\right] \end{aligned} \tag{18}$$

2.2. Case two: No random failure happens in uptime

In case two, we have $t \geq t_{1Z}$. Fig. 4 depicts the case two's stock level (in blue lines). It shows that when uptime ends, the stock level reaches H_1, and it climbs up to H_2 when the rework ends. Once the outsourced items are received, the stock level jumps to H before t_{3Z}. Fig. 5 displays the status of safety stock in case two, where it remains unchanged at all time since no failures occur. For the level of defective products in case two, one can refer to Fig. 3 but exclude the period of t_r. Similarly, we can observe the following relationships among parameters according to our model's assumption model description (see Fig. 4 to Fig. 6):

$$t_{1Z} = \frac{H_1}{P_{1A} - d_{1A} - \lambda} = \frac{Q\left(1-\pi\right)}{P_{1A}} \tag{19}$$

$$H_1 = \left(P_{1A} - d_{1A} - \lambda\right)t_{1Z} \tag{20}$$

$$t_{2Z} = \frac{x\left[\left(1-\pi\right)Q\right]}{P_{2A}} \tag{21}$$

$$H_2 = H_1 + \left(P_{2A} - \lambda\right)t_{2Z} \tag{22}$$

$$H = H_2 + \pi Q \tag{23}$$

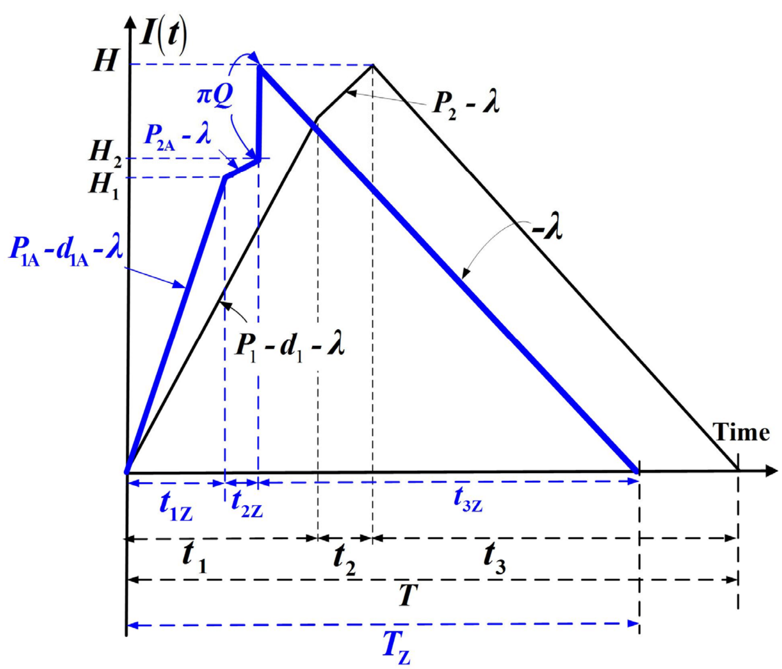

Fig. 4. The inventory level of the proposed study but with no machine failures (in blue lines) compared to a problem with only rework (in black thinner lines)

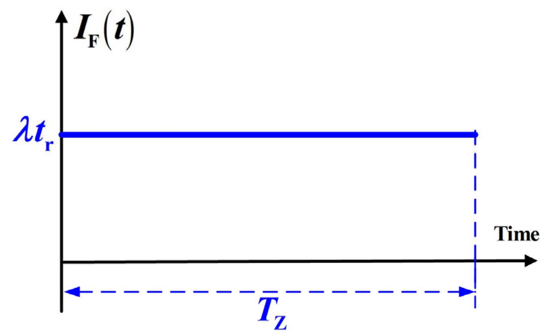

Fig. 5. The status of safety stock in case two

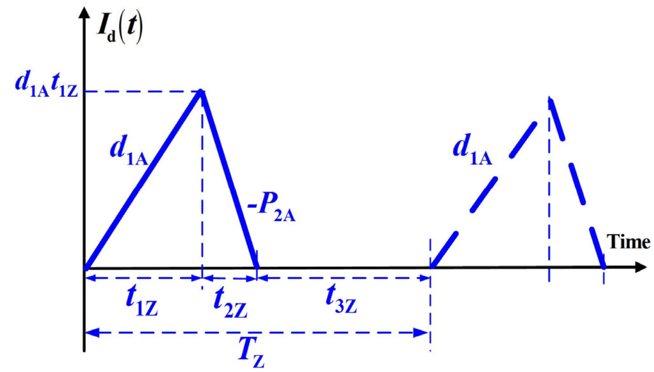

Fig. 6. The status of defective products in case two during T_Z

$$t_{3Z} = \frac{H}{\lambda} \tag{24}$$

$$T_Z = t_{1Z} + t_{2Z} + t_{3Z} \tag{25}$$

$TC(t_{1Z})_2$, the total cost per cycle in case two comprises the following: both the fixed and variable in-house production and outsourcing costs, safety stock relevant cost, rework cost, and holding costs (comprising the reworked items, perfect and defective products) during T_Z (see Eq. (26)).

$$\begin{aligned} TC(t_{1Z})_2 &= K_A + (1-\pi)QC_A + K_\pi + \pi QC_\pi(\ ) + (\lambda t_r)T_Z h_3 + x(1-\pi)QC_{RA} \\ &+ \frac{P_{2A}t_{2Z}}{2}(t_{2Z})h_1 + \left[(t_{1Z})\frac{H_1 + d_{1A}t_{1Z}}{2} + (t_{2Z})\frac{H_1+H_2}{2} + (t_{3Z})\frac{H}{2}\right]h \end{aligned} \tag{26}$$

Substituting Eqs. (1) to (8) in Eq. (26), we have $TC(t_{1Z})_2$ as follows:

$$\begin{aligned} TC(t_{1Z})_2 &= (1+\alpha_2)K + (1-\pi)Q(1+\alpha_3)C + (1+\beta_1)K + (\pi Q)(1+\beta_2)C \\ &+ (\lambda t_r)T_Z h_3 + x(1-\pi)Q(1+\alpha_3)C_R + \frac{\left[(1+\alpha_1)P_2\right]t_{2Z}}{2}(t_{2Z})h_1 \\ &+ \left[(t_{1Z})\frac{H_1 + x(1+\alpha_1)P_1 t_{1Z}}{2} + (t_{2Z})\frac{H_1+H_2}{2} + (t_{3Z})\frac{H}{2}\right]h \end{aligned} \tag{27}$$

2.3. Integration of cases 1 and 2, and the optimization procedure

In this study, we assume the Poisson-distributed failure rate β; so, the time to failure adheres to an Exponential-distributed rate with density function $f(t) = \beta e^{-\beta t}$ and cumulative density function $F(t) = (1 - e^{-\beta t})$. By utilizing the renewal reward theorem and applying the expected values of x for its random defectiveness, we have $E[TCU(t_{1Z})]$ as follows:

$$E\left[TCU(t_{1Z})\right] = \frac{\left\{\int_0^{t_{1Z}} E\left[TC(t_{1Z})_1\right]\cdot f(t)dt + \int_{t_{1Z}}^{\infty} E\left[TC(t_{1Z})_2\right]\cdot f(t)dt\right\}}{E[\boldsymbol{T}_Z]} \tag{28}$$

where $E[\boldsymbol{T}_Z]$, $E[T'_Z]$, and $E[T_Z]$ stand for the following:

$$E[\boldsymbol{T}_Z] = \int_0^{t_{1Z}} E[T'_Z]f(t)dt + \int_{t_{1Z}}^{\infty} E[T_Z]f(t)dt \tag{29}$$

$$E[T'_Z] = \frac{Q+\lambda t_r}{\lambda} = \frac{t_{1Z}P_{1A}\left[\frac{1}{(1-\pi)}\right] + \lambda t_r}{\lambda} \tag{30}$$

$$E[T_Z] = \frac{Q}{\lambda} = \frac{t_{1Z} P_{1A}\left[\frac{1}{(1-\pi)}\right]}{\lambda} \tag{31}$$

We first apply the $E[x]$ to formulas (18) and (27), and then substitute Eqs. (18), (27), and (29) in Eq. (28), $E[TCU(t_{1Z})]$ becomes as follows (see Appendix A for the detailed processes):

$$E\left[TCU\left(t_{1Z}\right)\right] = \frac{\lambda}{\frac{\lambda g\left(1-e^{-\beta t_{1Z}}\right)}{t_{1Z}\left(1+\alpha_1\right)P_1} + \frac{1}{(1-\pi)}} \begin{Bmatrix} \frac{W_1}{t_{1Z}} + W_2 + \left(t_{1Z}\right)W_3 + \frac{G_0}{t_{1Z}}\left(1-e^{-\beta t_{1Z}}\right) \\ -G_1\left(e^{-\beta t_{1Z}}\right) + \frac{G_2}{t_{1Z}}\left(1-e^{-\beta t_{1Z}}\right) + G_3\left(1-e^{-\beta t_{1Z}}\right) \end{Bmatrix} \tag{32}$$

Applying $E[TCU(t_{1Z})]$'s first-derivative and second-derivative, we gain formulas (A-5) and (A-6) (see Appendix A). Since, the 1st term on the right-hand side (RHS) of formula (A-6) is positive, then, $E[TCU(t_{1Z})]$ is convex if the second term on the RHS of formula (A-6) is also positive. Meaning that if $q(t_{1Z}) > t_{1Z} > 0$ (i.e., Eq. (A-7) holds). Upon verifying Eq. (A-7) is true, we solve t_{1Z}* by setting $E[TCU(t_{1Z})]$'s first-derivative = 0 (see Eq. (A-5)). Because the 1st term on the RHS of Eq. (A-5) is positive, so, we have the following:

$$\begin{Bmatrix} \left[W_3\left(v_1 - \lambda g\beta e^{-\beta t_{1Z}}\right) + G_1\left(\beta v_1 e^{-\beta t_{1Z}}\right) + G_3\left(\beta e^{-\beta t_{1Z}} v_1\right)\right] {t_{1Z}}^2 \\ +\left[-W_2\lambda g\beta e^{-\beta t_{1Z}} + 2\lambda g W_3\left(1-e^{-\beta t_{1Z}}\right) + \left(G_0+G_2\right)v_1\beta e^{-\beta t_{1Z}} + G_1\left(\beta\lambda e^{-\beta t_{1Z}} g\right)\right] t_{1Z} \\ + \begin{bmatrix} -W_1\left(\lambda g\beta e^{-\beta t_{1Z}} + v_1\right) + W_2\lambda g\left(1-e^{-\beta t_{1Z}}\right) - \left(G_0+G_2\right)v_1\left(1-e^{-\beta t_{1Z}}\right) \\ -G_1\lambda g\left(-e^{-2\beta t_{1Z}} + e^{-\beta t_{1Z}}\right) + G_3\lambda g\left(-2e^{-\beta t_{1Z}} + e^{-2\beta t_{1Z}} + 1\right) \end{bmatrix} \end{Bmatrix} = 0 \tag{33}$$

Let δ_2, δ_1, and δ_0 stand for the following:

$$\delta_2 = \left[W_3\left(v_1 - \lambda g\beta e^{-\beta t_{1Z}}\right) + G_1\left(\beta v_1 e^{-\beta t_{1Z}}\right) + G_3\left(\beta e^{-\beta t_{1Z}} v_1\right)\right]$$

$$\delta_1 = -W_2\lambda g\beta e^{-\beta t_{1Z}} + 2\lambda g W_3\left(1-e^{-\beta t_{1Z}}\right) + \left(G_0+G_2\right)v_1\beta e^{-\beta t_{1Z}} + G_1\left(\beta\lambda e^{-\beta t_{1Z}} g\right)$$

$$\delta_0 = \begin{bmatrix} -W_1\left(\lambda g\beta e^{-\beta t_{1Z}} + v_1\right) + W_2\lambda g\left(1-e^{-\beta t_{1Z}}\right) - \left(G_0+G_2\right)v_1\left(1-e^{-\beta t_{1Z}}\right) \\ -G_1\lambda g\left(-e^{-2\beta t_{1Z}} + e^{-\beta t_{1Z}}\right) + G_3\lambda g\left(-2e^{-\beta t_{1Z}} + e^{-2\beta t_{1Z}} + 1\right) \end{bmatrix}$$

Eq. (33) becomes as follows:

$$\delta_2\left(t_{1Z}\right)^2 + \delta_1\left(t_{1Z}\right) + \delta_0 = 0 \tag{34}$$

Lastly, by applying the square roots solution to Eq. (34), we gain the following t_{1Z}*:

$$t_{1Z}^* = \frac{-\delta_1 \pm \sqrt{\delta_1^2 - 4\delta_2\delta_0}}{2\delta_2} \tag{35}$$

*2.4. Searching algorithm for t_{1Z}**

Since the cumulative density function $F(t_{1Z}) = (1 - e^{-\beta t_{1Z}})$ has the values within [0, 1], so does $e^{-\beta t_{1Z}}$ – its complement. Rearrange Eq. (33), the following $e^{-\beta t_{1Z}}$ is gained:

$$e^{-\beta t_{1Z}} = \frac{-\begin{bmatrix} W_3 v_1 {t_{1Z}}^2 + 2\lambda g W_3 t_{1Z} - W_1\left(v_1\right) + W_2\lambda g \\ +\lambda g\left(G_1+G_3\right)\left(e^{-2\beta t_{1Z}}\right) - \left(G_0+G_2\right)v_1 + G_3\lambda g \end{bmatrix}}{\begin{Bmatrix} \left[-W_1\left(\lambda g\beta\right) - W_2\lambda g + \left(G_0+G_2\right)v_1\right] - \lambda g\left(G_1+2G_3\right) \\ +\left[-W_2\lambda g\beta - 2\lambda g W_3 + \left(G_0+G_2\right)v_1\beta + G_1\left(\beta\lambda g\right)\right]t_{1Z} \\ +W_3\left[-\lambda g\beta + \left(G_1+G_3\right)\left(\beta v_1\right)\right]{t_{1Z}}^2 \end{Bmatrix}} \tag{36}$$

Lastly, we propose the following algorithm to find t_{1Z}*:

(1) Let $e^{-\beta t_{1Z}} = 1$ and $e^{-\beta t_{1Z}} = 0$ for calculating Eq. (35) and gain the lower bound t_{1ZL} and upper bound t_{1ZU}.
(2) Utilize the present values of t_{1ZL} and t_{1ZU} to re-calculate a set of updated $e^{-\beta t_{1ZU}}$ and $e^{-\beta t_{1ZL}}$.
(3) Apply $e^{-\beta t_{1ZU}}$ and $e^{-\beta t_{1ZL}}$ to Eq. (35) again to obtain an updated bounds t_{1ZU} and t_{1ZL}.
(4) Verify if $t_{1ZU} = t_{1ZL}$ holds, if it does, then t_{1Z}* is derived (i.e., $t_{1ZU} = t_{1ZL} = t_{1Z}$*); otherwise, repeat the above-mentioned procedure (2), until ($t_{1ZU} = t_{1ZL}$) is true.

3. Numerical example

This section offers an example with the following simulated parameters' values (see Table 1) to show our model and the result's applicability.

Table 1
Parameters' values of our example

λ	M	K	C	P_2	β_2	C_R	α_1	C_1	g
4000	\$2500	\$200	\$2	5000	0.5	\$1	0.5	\$2	0.018
β	P_1	β_1	π	α_2	x	h	h_1	h_3	α_3
1	10000	-0.70	0.4	0.1	20%	\$0.4	\$0.4	\$0.4	0.1

*3.1. The E[TCU(t_{1Z})]'s convexity and t_{1Z}**

We first test for convexity of $E[TCU(t_{1Z})]$ (i.e., to test $q(t_{1Z}) > t_{1Z} > 0$ (see Appendix A: Eq. (A-7)). Since $e^{-\beta t_{1Z}}$ has the values within [0, 1], by initially assuming $e^{-\beta t_{1Z}} = 0$ and $e^{-\beta t_{1Z}} = 1$, and applying (35) we obtain $t_{1ZL} = 0.0674$ and $t_{1ZU} = 0.3455$. Then, calculating and testing Eq. (A-7) with $e^{-\beta t_{1ZL}}$ and $e^{-\beta t_{1ZU}}$, we gain respectively that $q(t_{1ZL}) = 0.1807 > t_{1ZL} > 0$ and $q(t_{1ZU}) = 0.4815 > t_{1ZU} > 0$. Hence, for $\beta = 1$, we ensure $E[TCU(t_{1Z})]$ is convex and the existence of optimal t_{1Z}*. A more comprehensive choice of β values are used to test for $E[TCU(t_{1Z})]$'s convexity to demonstrate our model's general usages, and the outcomes are exhibited in Table 2.

Table 2
The testing outcomes for $E[TCU(t_{1Z})]$'s convexity with various βs

β	$q(t_{1ZU})$	t_{1ZU}	$q(t_{1ZL})$	t_{1ZL}
10	1.1668	0.3395	0.0239	0.0115
7	0.7051	0.3398	0.0336	0.0161
4	0.4910	0.3405	0.0569	0.0267
3	0.4572	0.3411	0.0738	0.0340
2	0.4451	0.3422	0.1048	0.0460
1	**0.4815**	**0.3455**	**0.1807**	**0.0674**
0.5	0.5822	0.3521	0.2985	0.0843
0.01	2.8767	0.7569	2.2194	0.1065

Applying the searching algorithm (presented in section 2.4) to gain t_{1Z}* = 0.1115 and $E[TCU(t_{1Z}*)]$ = \$11,537. Its iterative outcomes are depicted in Table 3.

Table 3
Iterative outcomes of the searching algorithm for t_{1Z}*

Step	t_{1ZU}	$e^{-\beta t_{1ZU}}$	$E[TCU(t_{1ZU})]$	t_{1ZL}	$e^{-\beta t_{1ZL}}$	$E[TCU(t_{1ZL})]$	t_{1ZU} - t_{1ZL}
-	-	0	-	-	1	-	-
1	**0.3455**	0.7079	\$12109.19	**0.0674**	0.9348	\$11640.56	0.2781
2	0.1733	0.8409	\$11616.44	0.0958	0.9086	\$11546.28	0.0775
3	0.1307	0.8774	\$11547.30	0.1061	0.8993	\$11538.06	0.0246
4	0.1178	0.8889	\$11538.30	0.1097	0.8961	\$11537.20	0.0081
5	0.1136	0.8926	\$11537.23	0.1109	0.8950	\$11537.10	0.0027
6	0.1122	0.8939	\$11537.11	0.1113	0.8947	\$11537.09	0.0009
7	0.1117	0.8943	\$11537.09	0.1114	0.8946	\$11537.09	0.0003
8	0.1116	0.8944	\$11537.09	0.1115	0.8945	\$11537.09	0.0001
9	**0.1115**	0.8945	**\$11537.09**	**0.1115**	0.8945	**\$11537.09**	**0.0000**

Fig. 7 exhibits the collective effect of variations in runtime t_{1Z} and expedited rates factor α_1 on $E[TCU(t_{1Z})]$. It discloses that $E[TCU(t_{1Z})]$ upsurges as α_1 increases, and as t_{1Z} deviates both ways from the optimal t_{1Z}* (i.e., 0.1115), $E[TCU(t_{1Z})]$ knowingly rises.

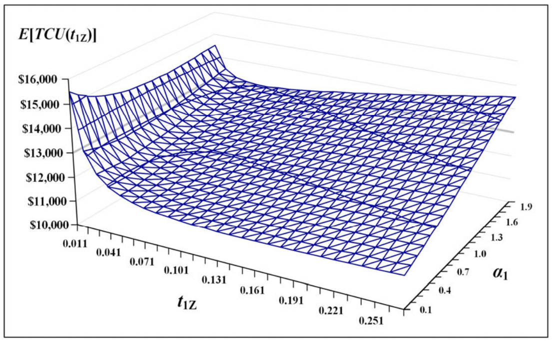

Fig. 7. The collective effect of variations in t_{1Z} and α_1 on $E[TCU(t_{1Z})]$

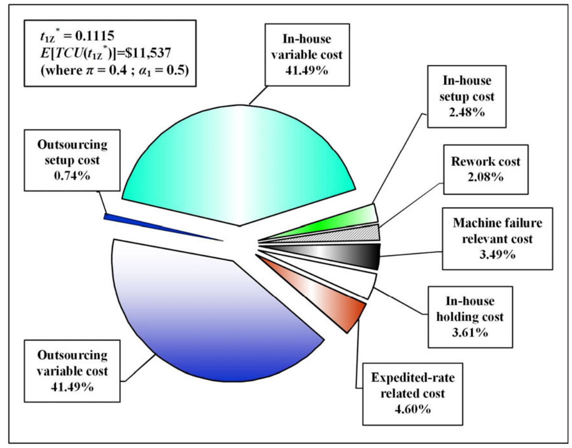

Fig. 8. The breakup of this example's $E[TCU(t_{1Z}*)]$

The proposed model can also provide detailed cost contributors of $E[TCU(t_{1Z}*)]$ as shown in Fig. 8. It reveals two critical cost contributors, i.e., the variable costs of outsourcing and in-house process, each evenly contributes 41.49% to $E[TCU(t_{1Z}*)]$. The system's quality cost includes a 3.49% relating to random machine failures and a 2.08% regarding defective products' reworking.

3.2. The impact of random machine failures and rework

Investigative results of the influence of random failures on $E[TCU(t_{1Z}*)]$ is displayed in Fig. 9. It shows $E[TCU(t_{1Z}*)]$ decreases as $1/\beta$ (i.e., mean-time-to-failure) rises. Notably, as $1/\beta$ surges to and beyond 0.20, $E[TCU(t_{1Z}*)]$ drops severely. Further investigation exposes a 3.34% increase in $E[TCU(t_{1Z}*)]$ due to the random machine failures.

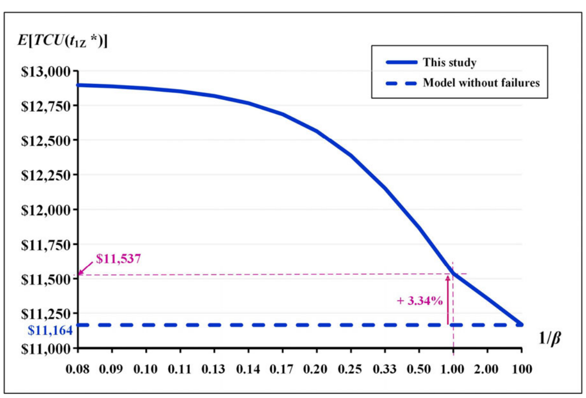

Fig. 9. The impact of changes in mean-time-to-failures on $E[TCU(t_{1Z}*)]$

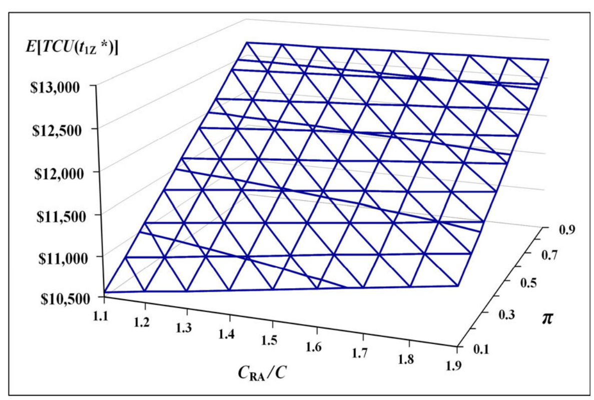

Fig. 10. The combined effect of (C_{RA} / C) and π on $E[TCU(t_{1Z}*)]$

Fig. 10 depicts the combined impact of variations in the (C_{RA} / C) and outsourcing portion π on $E[TCU(t_{1Z}*)]$. It indicates that $E[TCU(t_{1Z}*)]$ considerably surges as π increases, and it goes up both (C_{RA} / C) rises.

Fig. 11 discloses the collective impact of changes in π and x on total rework cost. It indicates that total rework cost drastically upsurges as x rises; but it drops as π increases.

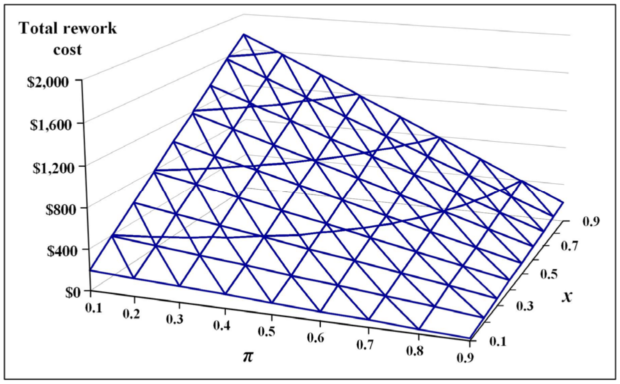

Fig. 11. The collective impact of changes in π and x on total rework cost

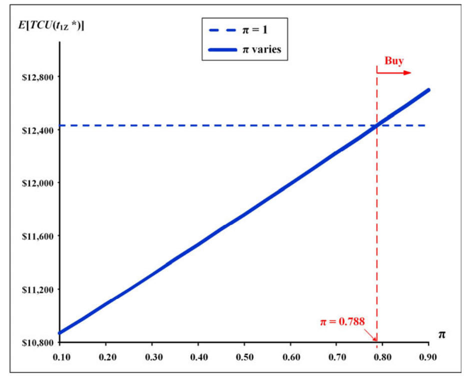

Fig. 12. The critical outsourcing portion π on the make-or-buy decision

3.3. The impact of dual uptime-reduction strategies

This study proposes dual uptime/utilization reduction strategies, and the following exploration results demonstrate our model's capability. Fig. 12 exposes the critical outsourcing portion π for making the make-or-buy decision. It shows as π increases to and over 0.788; clearly, the buy decision is beneficial. Fig. 13 exhibits the impact of the ratio (P_{1A} / P_1) (i.e., the expedited-rate versus standard rate) on utilization. As the ratio (P_{1A} / P_1) rises, utilization noticeably decreases. This example shows at (P_{1A} / P_1) = 1.5 (as our example assumes), utilization drops a 33.25% to 0.1915.

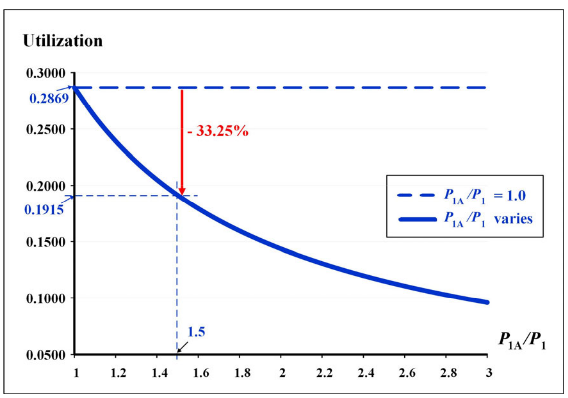

Fig. 13. The impact of (P_{1A} / P_1) on utilization

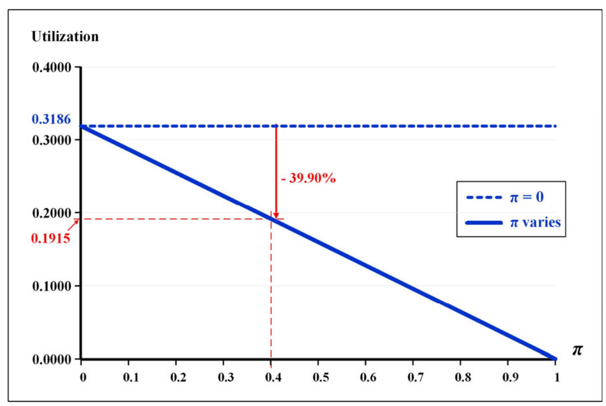

Fig. 14. The influences of variations in π on utilization

Fig. 14 illustrates the influences of changes in π on utilization. The machine utilization substantially decreases as π rises. This example shows at π = 0.4, utilization drops a 39.90% to 0.1915. For this example, at α_1 = 0.5 and π = 0.4, we further investigate the collective influence of changes in α_1 and π on $E[TCU(t_{1Z}*)]$. Fig. 15 discloses its analytical outcomes and reveals that starting with α_1 = 0.5 and increasing π is a more economical strategy to reduce utilization.

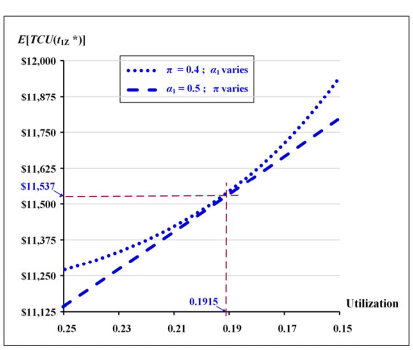

Fig. 15. The collective impact of differences in α_1 and π on $E[TCU(t_{1Z}*)]$

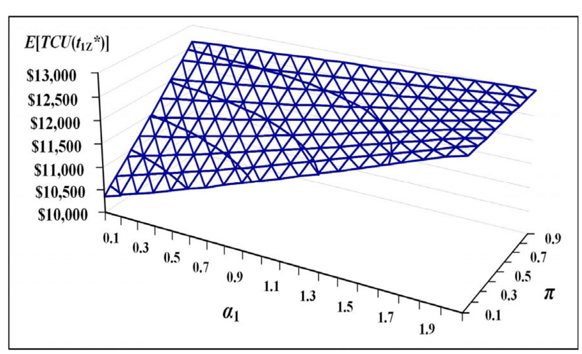

Fig. 16. The combined influence of α_1 and π on $E[TCU(t_{1Z}*)]$

Fig. 16 shows the combined influence of α_1 and π on $E[TCU(t_{1Z}*)]$. It reveals that $E[TCU(t_{1Z}*)]$ considerably increases as both α_1 and π surge. It also discloses π has more influence on $E[TCU(t_{1Z}*)]$'s upsurge than that of α_1. Fig. 17 compares the utilization of this study with that of existing works. It exposes that due to implementing dual uptime-reduction strategies, our utilization significantly declines a 33.3%, 39.9%, and 59.9% compared to the existing works, by paying the prices of a 3.89%, 9.17%, and 16.66% rise in $E[TCU(t_{1Z}*)]$, respectively. Specifically, $E[TCU(t_{1Z}*)]$ increases to \$11,537 from \$11,105, \$10,568, and \$9,890, respectively.

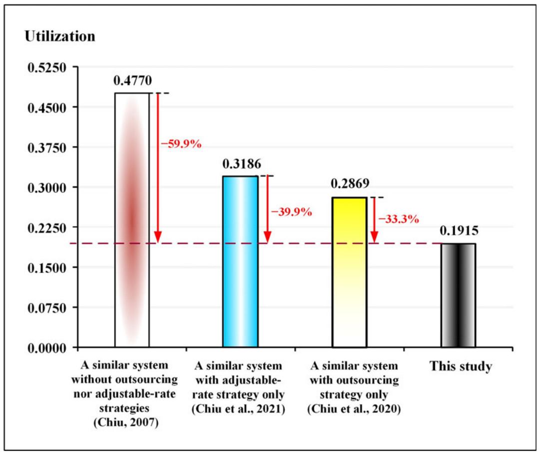

Fig. 17. Utilization comparison

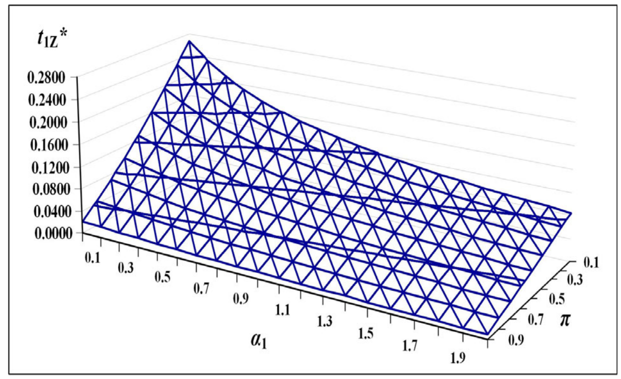

Fig. 18. The collective impact of α_1 and π on t_{1Z}*

Fig. 18 shows the collective impact of α_1 and π on t_{1Z}*. It discloses that t_{1Z}* noticeably decreases as both α_1 and π surge. It also reveals that π has more impact on runtime's drop than that of α_1.

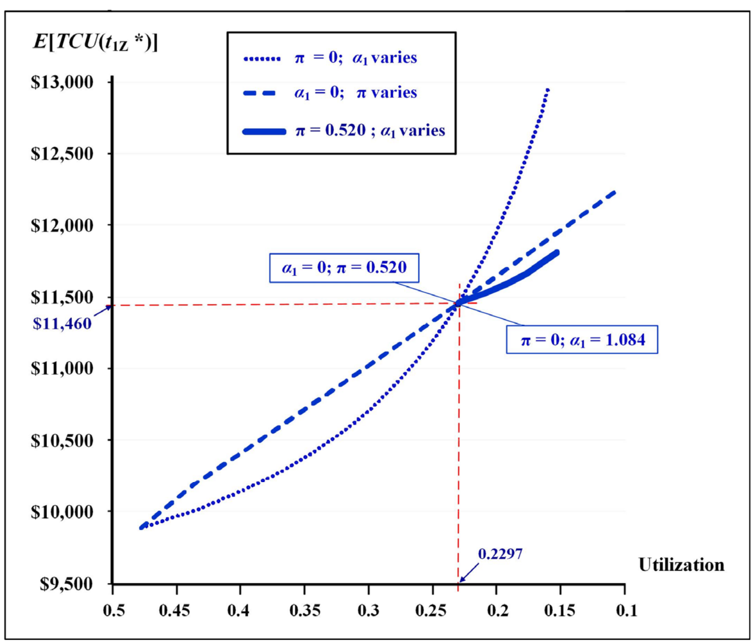

Fig. 19. Managerial insights regarding an effective/economic utilization-reduction strategy

4. Conclusions

The growing trend of rapid-response orders and inevitable production defects and failures have urged today's production managers to carefully evaluate these factors' effect on the production system's overall operating expenditures and potential operational disruptions. Inspired by helping them find the optimal runtime decision under these situations, this work develops a model (see subsection 2.1) featuring a partial outsourcing and expedited rate, rework of defects, and repairing failure machines to explore their collective effect on the problem's overall operating expenses. By using the techniques of model building and formulations, differential equations, and algorithms, we can thoroughly analyze the studied system, gain and minimize its overall expenses, and decide the optimal manufacturing runtime (refer to subsections 2.2 to 2.4). Lastly, we utilize numerical demonstrations to show the study's applicability and expose the following important, in-depth characteristics that facilitate managerial decision-making (see section 3):

(1) Confirmation of the study's applicability (see Table 2) and the convexity of $E[TCU(t_{1Z})]$ and its detailed contributors (refer to Table 3, and Figures 7 to 8);

(2) The impact and collective effect of stochastic failures, outsourcing factor, and rework of defects on $E[TCU(t_{1Z}*)]$ and total rework expenses (see Figures 9 to 11);

(3) The influence and combined influence of outsourcing and expedited rate factors on system utilization and $E[TCU(t_{1Z}*)]$ (see Figures 12 to 16);

(4) Comparing our utilization with existing studies and the collective impact of α_1 and π on $t_{1Z}*$ (refer to Figures 17 to 18);

(5) Managerial insights regarding an effective/economic utilization-reduction strategy (see Figure 19).

Examining the effect of random demand on the problem is a worthwhile research subject for the future.

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Appendix – A

The detailed derivations of $E[TCU(t_{1Z})]$ (i.e., Eq. (32)) and the proof of its convexity are exhibited as follows. We first apply the $E[x]$ to formulas (18) and (27), and then substitute formulas (18), (27), and (29) in formula (28), $E[TCU(t_{1Z})]$ becomes as follows:

$$E\left[TCU\left(t_{1Z}\right)\right]=\frac{\left\{\int_0^{t_{1Z}} E\left[TC\left(t_{1Z}\right)_1\right]\cdot f\left(t\right)dt+\int_{t_{1Z}}^{\infty} E\left[TC\left(t_{1Z}\right)_2\right]\cdot f\left(t\right)dt\right\}}{E[\boldsymbol{T}_Z]}$$

$$=\left[\frac{\lambda}{\dfrac{\lambda g\left(1-e^{-\beta t_{1Z}}\right)}{\left(t_{1Z}\right)\left(1+\alpha_1\right)P_1}+\dfrac{1}{\left(1-\pi\right)}}\right]\left[\begin{array}{l}\dfrac{\left(1+\beta_1\right)K}{t_{1Z}\left(1+\alpha_1\right)P_1}+\dfrac{\left(1+\alpha_2\right)K}{t_{1Z}\left(1+\alpha_1\right)P_1}\\ +\left[\left(1+\beta_2\right)C\left(\dfrac{\pi}{1-\pi}\right)+\left(1+\alpha_3\right)C+\left(1+\alpha_3\right)C_R E\left[x\right]\right]\\ +\left(t_{1Z}\right)\left\{\begin{array}{l}\dfrac{E\left[x\right]^2\left(1+\alpha_1\right)P_1}{2\left(1+\alpha_1\right)P_2}\left(h_1-h\right)\\ +\dfrac{h}{2\lambda}\dfrac{\left(1+\alpha_1\right)P_1}{\left(1-\pi\right)^2}\left\{\dfrac{\lambda E\left[x\right]\left(1-\pi\right)\left(-2\pi\right)}{\left(1+\alpha_1\right)P_2}-\dfrac{\lambda\left(1-\pi\right)\left(1+\pi\right)}{\left(1+\alpha_1\right)P_1}+1\right\}\end{array}\right\}\\ +\left[M+C_1\lambda g+C_T\lambda g+h_3\left(\dfrac{\lambda g^2}{2}\right)\right]\left(\dfrac{1-e^{-\beta t_{1Z}}}{t_{1Z}\left(1+\alpha_1\right)P_1}\right)+\dfrac{h_3 g}{1-\pi}\left(1-e^{-\beta t_{1Z}}\right)\\ +\dfrac{\left[hg\left(\left(1+\alpha_1\right)P_1-\lambda\right)+h_3\lambda g\right]}{t_{1Z}\left(1+\alpha_1\right)P_1}\left(-t_{1Z}e^{-\beta t_{1Z}}-\dfrac{1}{\beta}e^{-\beta t_{1Z}}+\dfrac{1}{\beta}\right)\end{array}\right] \tag{A-1}$$

Let W_1, W_2, W_3, G_0, G_1, G_2, and G_3 be the following:

$$\begin{aligned} W_1&=\frac{\left(1+\beta_1\right)K}{\left(1+\alpha_1\right)P_1}+\frac{\left(1+\alpha_2\right)K}{\left(1+\alpha_1\right)P_1}\\ W_2&=\left[\left(1+\beta_2\right)C\left(\frac{\pi}{1-\pi}\right)+\left(1+\alpha_3\right)C+\left(1+\alpha_3\right)C_R E\left[x\right]\right]\\ W_3&=\frac{E\left[x\right]^2\left(1+\alpha_1\right)P_1\left(h_1-h\right)}{2\left(1+\alpha_1\right)P_2}+\frac{h}{2\lambda}\frac{\left(1+\alpha_1\right)P_1}{\left(1-\pi\right)^2}\left\{\frac{\lambda E\left[x\right]\left(1-\pi\right)\left(-2\pi\right)}{\left(1+\alpha_1\right)P_2}-\frac{\lambda\left(1-\pi\right)\left(1+\pi\right)}{\left(1+\alpha_1\right)P_1}+1\right\}\end{aligned} \tag{A-2}$$

$$\begin{aligned} G_0&=\frac{M+C_1\lambda g+C_T\lambda g+h_3\left(\dfrac{\lambda g^2}{2}\right)}{\left(1+\alpha_1\right)P_1};\quad G_1=\frac{\left[hg\left(\left(1+\alpha_1\right)P_1-\lambda\right)+h_3\lambda g\right]}{\left(1+\alpha_1\right)P_1}\\ G_2&=\frac{\left[hg\left(\left(1+\alpha_1\right)P_1-\lambda\right)+h_3\lambda g\right]}{\left(1+\alpha_1\right)P_1\beta};\quad G_3=\frac{h_3 g}{1-\pi}.\end{aligned} \tag{A-3}$$

Then, we can rearrange $E[TCU(t_{1Z})]$ as follows:

$$E\left[TCU\left(t_{1Z}\right)\right]=\frac{\lambda}{\dfrac{1}{(1-\pi)}+\dfrac{\lambda g\left(1-e^{-\beta t_{1Z}}\right)}{t_{1Z}\left(1+\alpha_1\right)P_1}}\cdot\left\{\begin{array}{l}\dfrac{W_1}{t_{1Z}}+W_2+\left(t_{1Z}\right)W_3+\dfrac{G_0}{t_{1Z}}\left(1-e^{-\beta t_{1Z}}\right)-G_1\left(e^{-\beta t_{1Z}}\right)\\+\dfrac{G_2}{t_{1Z}}\left(1-e^{-\beta t_{1Z}}\right)+G_3\left(1-e^{-\beta t_{1Z}}\right)\end{array}\right\} \tag{32}$$

Let v_1 be the following:

$$\frac{\left(1+\alpha_1\right)P_1}{(1-\pi)} \tag{A-4}$$

Applying $E[TCU(t_{1Z})]$'s the first--derivative and second-derivative, we gain formulas (A-5) and (A-6) below:

$$\frac{dE\left[TCU\left(t_{1Z}\right)\right]}{d\left(t_{1Z}\right)}=\frac{\lambda\left(1+\alpha_1\right)P_1}{\left[t_{1Z}v_1+\lambda g\left(1-e^{-\beta t_{1Z}}\right)\right]^2}\left\{\begin{array}{l}-W_1\left[\lambda g\beta e^{-\beta t_{1Z}}+v_1\right]-W_2\lambda g\left(\beta e^{-\beta t_{1Z}}t_{1Z}+e^{-\beta t_{1Z}}-1\right)\\+W_3\left[t_{1Z}{}^2v_1-\lambda g\beta e^{-\beta t_{1Z}}t_{1Z}{}^2-2\lambda g e^{-\beta t_{1Z}}t_{1Z}+2\lambda g t_{1Z}\right]\\+\left(G_0+G_2\right)v_1\left(\beta e^{-\beta t_{1Z}}t_{1Z}+e^{-\beta t_{1Z}}-1\right)\\-G_1\left(-\beta v_1 e^{-\beta t_{1Z}}t_{1Z}{}^2-\beta\lambda e^{-\beta t_{1Z}}t_{1Z}g-\lambda e^{-2\beta t_{1Z}}g+\lambda e^{-\beta t_{1Z}}g\right)\\+G_3\left(\beta e^{-\beta t_{1Z}}t_{1Z}{}^2v_1-2\lambda e^{-\beta t_{1Z}}g+\lambda g e^{-2\beta t_{1Z}}+\lambda g\right)\end{array}\right\} \tag{A-5}$$

$$\frac{d^2E\left[TCU\left(t_{1Z}\right)\right]}{d\left(t_{1Z}\right)^2}=\frac{\lambda\left(1+\alpha_1\right)P_1}{\left[t_{1Z}v_1+\lambda g\left(1-e^{-\beta t_{1Z}}\right)\right]^3}\left\{\begin{array}{l}W_1\left(\beta^2e^{-2\beta t_{1Z}}\lambda^2g^2+\beta^2e^{-\beta t_{1Z}}\lambda^2g^2+4\beta e^{-\beta t_{1Z}}v_1\lambda g+2\left[v_1\right]^2+\beta^2e^{-\beta t_{1Z}}t_{1Z}\lambda g v_1\right)\\+W_2\lambda g\begin{pmatrix}2e^{-\beta t_{1Z}}v_1-2v_1+2\beta e^{-2\beta t_{1Z}}\lambda g-2\beta e^{-\beta t_{1Z}}\lambda g\\+\beta^2e^{-\beta t_{1Z}}t_{1Z}{}^2v_1+2\beta e^{-\beta t_{1Z}}t_{1Z}v_1+\beta^2e^{-2\beta t_{1Z}}\lambda g t_{1Z}+\beta^2e^{-\beta t_{1Z}}\lambda g t_{1Z}\end{pmatrix}\\+W_3\lambda g\begin{pmatrix}2\lambda g+2e^{-2\beta t_{1Z}}\lambda g-4e^{-\beta t_{1Z}}\lambda g+\beta^2e^{-2\beta t_{1Z}}t_{1Z}{}^2\lambda g+\beta^2e^{-\beta t_{1Z}}t_{1Z}{}^2\lambda g\\+4\beta e^{-2\beta t_{1Z}}t_{1Z}\lambda g-4\beta e^{-\beta t_{1Z}}t_{1Z}\lambda g+\beta^2e^{-\beta t_{1Z}}t_{1Z}{}^3v_1\end{pmatrix}\\-v_1\left(G_0+G_2\right)\begin{pmatrix}2\beta\lambda e^{-2\beta t_{1Z}}g+2e^{-\beta t_{1Z}}v_1-2\beta\lambda e^{-\beta t_{1Z}}g+\beta^2e^{-\beta t_{1Z}}t_{1Z}{}^2v_1\\-2v_1+\beta^2\lambda e^{-2\beta t_{1Z}}t_{1Z}g+\beta^2\lambda e^{-\beta t_{1Z}}t_{1Z}g+2\beta e^{-\beta t_{1Z}}t_{1Z}v_1\end{pmatrix}\\-G_1e^{-2\beta t_{1Z}}\begin{pmatrix}2\beta\lambda^2g^2-2\beta\lambda^2e^{\beta t_{1Z}}g^2+2\lambda g v_1-2\lambda e^{\beta t_{1Z}}g v_1+\beta^2\lambda^2g^2t_{1Z}\\+\beta^2\lambda^2e^{\beta t_{1Z}}g^2t_{1Z}+4\beta\lambda g t_{1Z}v_1+\beta^2\lambda g t_{1Z}{}^2v_1\\+2\beta^2\lambda e^{\beta t_{1Z}}g t_{1Z}{}^2v_1-2\beta\lambda e^{\beta t_{1Z}}g t_{1Z}v_1+\beta^2e^{\beta t_{1Z}}t_{1Z}{}^3\left[v_1\right]^2\end{pmatrix}\\-v_1G_3\begin{pmatrix}2\lambda g+2\lambda e^{-2\beta t_{1Z}}g-4\lambda e^{-\beta t_{1Z}}g+4\beta\lambda e^{-2\beta t_{1Z}}t_{1Z}g+\beta^2\lambda e^{-2\beta t_{1Z}}t_{1Z}{}^2g\\+\beta^2\lambda e^{-\beta t_{1Z}}t_{1Z}{}^2g-4\beta\lambda e^{-\beta t_{1Z}}t_{1Z}g+\beta^2e^{-\beta t_{1Z}}t_{1Z}{}^3v_1\end{pmatrix}\end{array}\right\} \tag{A-6}$$

As the 1st term on the RHS of formula (A-6) is positive, if the second term of RHS of Eq. (A-6) is also positive, then, $E[TCU(t_{1Z})]$ is convex. Meaning that if the following $q(t_{1Z}) > t_{1Z} > 0$ holds.

$$q\left(t_{1z}\right)=\frac{\left\{\begin{array}{l}W_1\left(\beta^2e^{-2\beta t_{1Z}}\lambda^2g^2+\beta^2e^{-\beta t_{1Z}}\lambda^2g^2+4\beta e^{-\beta t_{1Z}}v_1\lambda g+2v_1{}^2\right)\\+W_2\lambda g\left(2e^{-\beta t_{1Z}}v_1-2v_1+2\beta e^{-2\beta t_{1Z}}\lambda g-2\beta e^{-\beta t_{1Z}}\lambda g\right)\\+W_3\lambda g\left(2\lambda g+2e^{-2\beta t_{1Z}}\lambda g-4e^{-\beta t_{1Z}}\lambda g\right)\\-v_1\left(G_0+G_2\right)\left(2\beta\lambda e^{-2\beta t_{1Z}}g+2e^{-\beta t_{1Z}}v_1-2\beta\lambda e^{-\beta t_{1Z}}g-2v_1\right)\\-G_1e^{-2\beta t_{1Z}}\left(2\beta\lambda^2g^2-2\beta\lambda^2e^{\beta t_{1Z}}g^2+2\lambda g v_1-2\lambda e^{\beta t_{1Z}}g v_1\right)\\-v_1G_3\left(2\lambda g+2\lambda e^{-2\beta t_{1Z}}g-4\lambda e^{-\beta t_{1Z}}g\right)\end{array}\right\}}{-\left\{\begin{array}{l}W_1\left(\beta^2e^{-\beta t_{1Z}}v_1\lambda g\right)+W_2\lambda g\left(\beta^2e^{-\beta t_{1Z}}v_1t_{1Z}+2\beta e^{-\beta t_{1Z}}v_1+\beta^2e^{-2\beta t_{1Z}}\lambda g+\beta^2e^{-\beta t_{1Z}}\lambda g\right)\\+W_3\lambda g\left(\beta^2e^{-2\beta t_{1Z}}t_{1Z}\lambda g+\beta^2e^{-\beta t_{1Z}}t_{1Z}\lambda g+4\beta e^{-2\beta t_{1Z}}\lambda g-4\beta e^{-\beta t_{1Z}}\lambda g+\beta^2e^{-\beta t_{1Z}}t_{1Z}{}^2v_1\right)\\-v_1\left(G_0+G_2\right)\left(\beta^2e^{-\beta t_{1Z}}t_{1Z}v_1+\beta^2\lambda e^{-2\beta t_{1Z}}g+\beta^2\lambda e^{-\beta t_{1Z}}g+2\beta e^{-\beta t_{1Z}}v_1\right)\\-G_1e^{-2\beta t_{1Z}}\begin{pmatrix}\beta^2\lambda^2g^2+\beta^2\lambda^2e^{\beta t_{1Z}}g^2+4\beta\lambda g v_1+\beta^2\lambda g t_{1Z}v_1\\+2\beta^2\lambda e^{\beta t_{1Z}}g t_{1Z}v_1-2\beta\lambda e^{\beta t_{1Z}}g v_1+\beta^2e^{\beta t_{1Z}}t_{1Z}{}^2v_1{}^2\end{pmatrix}\\-v_1G_3\left(4\beta\lambda e^{-2\beta t_{1Z}}g+\beta^2\lambda e^{-2\beta t_{1Z}}t_{1Z}g+\beta^2\lambda e^{-\beta t_{1Z}}t_{1Z}g-4\beta\lambda e^{-\beta t_{1Z}}g+\beta^2e^{-\beta t_{1Z}}t_{1Z}{}^2v_1\right)\end{array}\right\}}>t_{1z}>0 \tag{A-7}$$