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The opportunistic newsvendor problem: Defining the optimal purchase quantity of resalable items, whose value may appreciate

Francesco Zammori[a*], Giovanni Romagnoli[a] and Serena Filippelli[a]

[a]*Department of Engineering and Architecture, University of Parma, Parco Area delle Scienze 181/A, Parma, Italy*

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A B S T R A C T

With Newsvendor Problem (NvP) we refer to a specific class of inventory management problems, valid for a single item with stochastic demand over a single period. In the standard version, the newsvendor is allowed to issue a single order, before he or she can observe the actual demand. Since the newsvendor can face both overage and underage costs, due to lost sales or residual stock, the objective is to define the optimal order size that maximizes the expected profit. In this paper, we consider a specific version of the NvP, in which the buyer has the opportunity to make a last and single order for opportunistic reasons. Specifically, we consider discontinued, collectible items, for which demand will not vanish and whose value might appreciate. Hence, the objective is to define the optimal quantity that should be purchased, just before the item is retired from the market or sold-out, and that should be sold as soon as the price rises over a predefined target level. An optimal solution, maximizing the expected profit, is obtained both in case of negligible and non-negligible stockholding costs. In the latter case, to obtain the optimal solution in implicit form, some simplifying assumptions are needed. Hence, a thorough numerical analysis is finally performed, as a way to empirically demonstrate both the robustness and the accuracy of the model, in several scenarios differentiated in terms of costs and customers' demand.

1. Introduction

To emerge in a competitive marketplace, entrepreneurship, business acumen and experience do no longer suffice. To make optimal decisions, managers should always ponder their choices leveraging both on their market knowledge and on decisional models with a sound mathematical basis. Only in this way, in fact, it is possible to carry out what-if analyses and envisage critical issues that might be overlooked if decisions were made based only on one's own expertise. For long term strategic decisions multi-criteria models are generally used (Zopounidis & Doumpos, 2002), whereas in case of operational decisions, optimization models are generally preferred (Murty, 2010). The Newsvendor problem (NvP) is a notable example of such optimization models, in the supply chain management field. Indeed, it has long been used as a way to determine optimal purchase quantities and inventory levels (Arrow et al., 1951). The classic NvP operates in a single-period single-product context (Khouja, 1999), where a newsboy must decide how many newspapers to order for subsequent resale. The newsboy represents the company, while the newspapers represent a perishable item that the company buys, or manufactures, for resale. Demand is random and the company does not know the exact quantity that will be sold; consequently, in case of extra demand some sales are lost, whereas in case of excess inventory some items remain unsold and must be disposed of. The objective is to choose the optimal order quantity that maximizes the expected profit.

* Corresponding author.
E-mail address: francesco.zammori@unipr.it (F. Zammori)

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Despite its simplicity, managerial insights provided by the NvP are many and can lead to a wide range of applications in several industrial fields, such as airline, hospitality and fashion goods industries (Choi, 2012). Literature in the subject matter is therefore extensive, as many variations to the basic model have been proposed and discussed so far. Excellent reviews and classifications can be found in the comprehensive works by Qin et al. (2011) and by Turken et al. (2012). Briefly, the main variations, generally referred to as the extended newsvendor model, include additional constraints and/or assumptions such as: *(i)* limited or random production capacity (Wu et al. 2013; Shi et al. 2020), *(ii)* promotions, quantity discounts and pricing policies to coordinate the supply chain (Jadidi et al., 2021), *(iii)* multiple products (Zhang and Du, 2010) and *(iv)* multiple periods (Kim et al., 2015). Some authors have even considered demand dependent selling price (Ullah et al. 2019) and/or stock dependent demand (Dana & Petruzzi, 2001; Urban, 2005). In the latter case, demand is assumed to be a function either of the initial or the instantaneous stock level and may thus vary over time. Other two common extensions are the risk-averse (Agrawal and Shadri, 2000; Arikan and Fichtinger, 2017) and the competitive newsvendor problem (Lippman and McCardle, 1997). In the first case, the decision maker's attitude towards risk is explicitly considered in the optimization model, whereas, in the second case, the monopolist vendor is replaced by a set of similar vendors competing in the same market. In this case competition affects both inventories and pricing because, in case of excess demand at a certain vendor, a portion of the unsatisfied buyers may be tempted to buy at another vendor. Hence, the objective is to define the split of demands (among the competing vendors) that maximizes the overall profit and to find possible equilibrium states. It is interesting to note that recently these extensions are jointly considered, as in the work by Wu et al. (2014) and Wu et al. (2018).

What is important to stress, is the fact that, in all its formulations, the NvP considers perishable or seasonal products that can be sold within a limited period of time, after which they lose some or all of their initial value and must be underpriced or even disposed of. Products that progressively lose their economic value in time are very common, but in some market niches an opposite pattern may be observed as well, and price may go up rather than down, over time. Although this condition is not common, it may arise for discontinued products whose market demand has not vanished, such as collectible items or the like, artistic furnishing objects, art memorabilia (Burton and Jacobsen, 1999; Matheson and Baade, 2004). In this peculiar condition, the objective should be to define the optimal quantity to be opportunistically purchased, just before the item is retired from the market, and sold as soon as demand and price rise over a predefined target level. How to do so is the objective and the main contribution of this paper. Specifically, an optimal solution, maximizing the expected profit, is obtained both in case of negligible and non-negligible stock holding costs. In the latter case, to obtain the optimal solution in implicit form, some simplifying assumptions are needed. Hence, a thorough numerical analysis is finally presented, to empirically demonstrate both the robustness and the accuracy of the model, in several scenarios differentiated in terms of costs and customers' demand.

2. The Opportunistic Newsvendor Problem (ONvP)

As noted in the introduction section, the NvP considers perishable products that can be sold at a profit only for a limited period of time. After that, the unsold products lose part of their initial value, and must be underpriced or even disposed of, if all the initial value has been lost. In this section, instead, we reformulate the NvP to consider the peculiar case of products whose value may increase, rather than decrease. This alternative version of the NvP, from here on referred to as the Opportunistic Newsvendor problem (ONvP) is provided in the following sub-sections, for different items and demand patterns. For convenience, the full notation of all the developed models can be found in Appendix A.

2.1 The basic ONvP

At first, we consider the case of products for which stock holding costs can be neglected. This condition typically applies to non-bulky and non-perishable products with a rather low purchasing price, but also if the resale takes place shortly after the opportunistic purchase. In all these cases, stock holding costs might be neglected. A possible example is that of tickets for an event (such as a concert, an exhibition, a sport match, etc.) that will take place at a certain date, say t_2. The official distribution service sells tickets at the standard price p_0, for a certain period, say from t_0 to t_1, when sales close. Note that the time $t_1 \leq t_2$ may be a predefined date or may be the time when tickets sell out. Anyhow, if the demand is higher than the offer, it is possible that after t_1 the price will rise, providing that resale is legal. To take advantage of this fact, an opportunistic buyer could buy at price p_0 a batch of tickets q, and should try to resell all of them, from t_1 to t_2, at a higher price p_h. Obviously, the risk faced by the buyer is that demand is lower than the offer, or that by any other reason the tickets price decreases, so that he or she has to resell the tickets at a price p_l lower than the original one, or even dispose of them if the price falls to zero.

To summarize, sales are divided into two periods. In the first one, of length $T_1 = (t_1 - t_0)$, the opportunistic buyer purchases tickets at the standard price p_0, hoping to resell them at a higher price in period two, of length $T_2 = (t_2 - t_1)$. In T_2, indeed, two mutually exclusive re-selling prices, are possible. The first one p_h is an optimistic price higher than p_0, while the latter one p_l is a pessimistic price lower than p_0.

We also assume that:

- the probability of p_h is $\alpha > 0.5$ and that of p_l is $(1-\alpha)$,
- the demand x in T_2 is a random variable, with probability distribution $f(x)$,
- the probability density function (p.d.f.) $f(x)$ is the same whether the re-selling price is p_h or p_l (i.e., $f(x)$ does not depend on the price's pattern).

So, letting q be the quantity purchased by the buyer, the profit π depends on the price patterns, that is:

$$\begin{aligned} \pi &= \min(q,x)\cdot\ p_h - \ p_0 q, \quad \textit{with probability}\ \alpha \\ \pi &= \min(q,x)\cdot\ p_l - \ p_0 q, \quad \textit{with probability}\ (1-\alpha) \end{aligned} \tag{1}$$

And the expected profit becomes:

$$\begin{aligned} E(\pi) &= \ \alpha p_h E[\min(q,x)] + (1-\alpha)\cdot p_l E[\min(q,x)] - \ p_0 q \\ &= \bar{p} E[\min(q,x)] - \ p_0 q \\ &= \bar{p} E[x-(x-q)^+] - \ p_0 q = \bar{p}\mu_x - \bar{p} E[(x-q)^+] - p_0 q \end{aligned} \tag{2}$$

where:

- $\mu_x = E[x]$
- $\bar{p} = \alpha p_h + (1-\alpha)\cdot p_l$,
- $(x-q)^+ = \max\{0, x-q\}$.

Now, making use of the fact that that $\max\{0, x-q\}$ can be expressed as $\int_q^{+\infty}(x-q)f(x)dx$, we finally have:

$$E(\pi) = \ \bar{p}\mu_x - \bar{p}\int_q^{+\infty}(x-q)f(x)dx - p_0 q \tag{3}$$

This function is convex (it is easy to see that its second derivative is always negative) and so, to find its maximum, we simply equate to zero the first derivative with respect to q:

$$\frac{dE[\pi]}{dq} = \ \bar{p}\int_q^{+\infty} f(x)dx - p_0 = \bar{p}\,[1-F(q)] - p_0 = 0 \tag{4}$$

Solving for q the optimal quantity q^* is finally obtained:

$$q^* = F^{-1}\left(1 - \frac{p_0}{\bar{p}}\right) \tag{5}$$

where $F(x)$ is the cumulative distribution function (c.d.f.) and $F^{-1}(x)$ its inverse. Note that, since $F(x)$ is defined on the interval [0; 1], p_0 must be less than $\bar{p}$ in Eq. (5). This is correct since the vendor would have no incentive to make an opportunistic purchase if the expected sale price were equal to the purchase one. Also note that, for a symmetric distribution function, $q^* = \mu_x$ if $\bar{p} = 2p_0$.

2.2 Assumptions relaxation: continuous distribution of the reselling price

The model can be easily extended to the case of a reselling price that, rather than being limited to two mutually exclusive values, follows a certain distribution function $g(y)$. In this case, letting x and y denoting the demand and the reselling prince in T_2, the profit becomes:

$$\pi = \min(q,x)\cdot\ y - \ p_0 q \tag{6}$$

And the expected value can be computed as in (7):

$$\begin{aligned} E[\pi] &= \iint_0^{+\infty} [(x-(x-q)^+)\cdot y - qp_0]\ \cdot f_{x,y}(x,y)dxdy \\ &= \int_0^{+\infty}\left(\int_0^{+\infty}[(x-(x-q)^+)\cdot y - qp_0]f(y)dy\right)f(x)dx - qp_0 \end{aligned} \tag{7}$$

$$= \int_0^{+\infty} \left[(x - (x-q)^+) \cdot \mu_y \right] f(x)dx - qp_0 = \mu_y \mu_x - \mu_y \int_q^{+\infty} (x-q)f(x)dx - qp_0$$

where $f_{x,y}(x,y)$ is the conjoint distribution of x and y, and we took advantage of the independence of $f(x)$ and $f(y)$. Taking the derivative with respect to q and equating to zero, the optimum purchasing quantity q^* becomes:

$$q^* = F^{-1}\left(1 - \frac{p_0}{\mu_y}\right) \quad (8)$$

In total agreement with Eq. (5).

2.3 Assumptions relaxation: reselling price depending on the demand of period two

A more interesting case emerges when the demand and the reselling price in T_2 are strictly related, so that if demand in T_2 is low also the reselling price will be low and vice versa. We can formalize this concept by stating that a certain range of the reselling price $(p_i, p_{(i+1)}]$ corresponds to a certain distribution $f_{(i+1)}(x)$ of the demand. More precisely, we have that the conjoint distribution of x and y becomes:

$$f_{x,y} = \begin{cases} f_1(x)g(y), & for\ y \in [p_0, p_1) \\ f_2(x)g(y), & for\ y \in [p_1, p_2) \\ & \dots \\ f_n(x)g(y), & for\ y \in [p_{(n-1)}, p_n) \end{cases} \quad (9)$$

So we have that:

$$E[\pi] = \iint_0^{+\infty} [(x - (x-q)^+) \cdot y - qp_0] \cdot f_{x,y}(x,y)dxdy$$

$$= \sum_{i=1}^{n} \int_0^{+\infty} \left(\int_{p_{(i-1)}}^{p_i} [(x-(x-q)^+) \cdot y - qp_0] g(y)dy \right) f_i(x)dx \quad (10)$$

$$= -qp_0 + \sum_{i=1}^{n} \bar{y}_i \cdot \left(\mu_{x_i} - \int_q^{+\infty} (x-q)f_i(x)dx \right)$$

where:

- $\bar{y}_i = \int_{p_{(i-1)}}^{p_i} yg(y)dy = E\left[y | y \in [p_{(i-1)}, p_i)\right] \cdot [G(p_i) - G(p_{i-1})]$,
- $\mu_{x_i} = \int_0^{+\infty} x\, f_i(x)dx$.

If $g(y)$ is discrete, letting α_i the probability that the price equals p_i, we have:

$$E[\pi] = \sum_{i=1}^{n} \alpha_i p_i \cdot \left(\mu_{x_i} - \int_q^{+\infty} (x-q)^+ f_i(x)dx \right) - qp_0 \quad (11)$$

And the minimum is found when:

$$\frac{dE[\pi]}{dq} = \sum_{i=1}^{n} \alpha_i p_i \cdot [1 - F_i(q)] - q = 0 \quad (12)$$

which, de facto, is a generalization of Eq. (4).

3. The ONvP with stockholding costs

In the standard ONvP, we have implicitly neglected the holding (or opportunity) costs incurred by the buyer. Very frequently, in fact, both T_1 and T_2 are short and $T_2 \ll T_1$ as in the case of the official ticket distribution lasting for a few weeks and the resale starting immediately afterwards and ending within a few days. Sometimes, however, tickets are sold well in advance of the event's scheduled date, as for the final of a football tournament or an Olympics game. In this case the buyer anticipates (and risks) a sum hoping to get a profit in the future and so, the ONvP cannot neglect the cost of the invested capital.

The same condition also occurs for discontinued products whose market demand has not vanish, as for collectible items such as building kits (e.g. Lego ©), modelling (e.g., rail transport, ships, cars, airplanes, etc.), comics, or the like. Also in this case resale will not be almost instantaneous, but will take place in the future. So, since both T_1 and T_2 are rather long and, presumably, $T_2 > T_1$, holding or opportunity costs and eventually a disposal cost (to get rid of the excessive inventory) cannot be neglected and should be included in the optimization process.
The disposal cost C is incurred whenever the purchased quantity q is higher of the actual demand x. Hence, letting c $\left[\frac{euro}{item}\right]$ be the unit disposal cost, we have:

$$C = c \cdot max(0, q - x) = c \cdot (q - x)^+ \tag{13}$$

The holding cost H, instead, is always sustained by the buyer. Specifically, since in T_1 the buyer does not sell any products, the purchased quantity q remains unaltered until T_2, when it starts decreasing because of the sales. So, we have that:

$$H = hqT_1 + h\bar{s}T_2 = h \cdot (qT_1 + \bar{s}T_2) \tag{14}$$

where $h\left[\frac{euro}{time\ unit \cdot item}\right]$ is the unit holding cost per unit of time, and $\bar{s}$ is the average stock in T_2. As Eq. (14) shows, H is made by a fixed and by a random component; the latter one is a straight consequence of the random demand in T_2. Also, depending on the ratio (q/x), two alternative cases must be considered because if $(q/x) \leq 1$, the purchased quantity ends before the end of T_2, whereas if $(q/x) > 1$ the opportunistic buyer is left with an extra stock that must be disposed of at the end of period T_2. Similarly to what was done by Shaolong et al. (2018), to compute the average stock $\bar{s}$, we assume a linear consumption of the stock, as graphically shown in Fig. 1 and Fig. 2. Specifically, as Fig. 1 shows, if $q \geq x$ the stock s linearly decreases from the initial quantity q, to the residual one $(q - x)$.

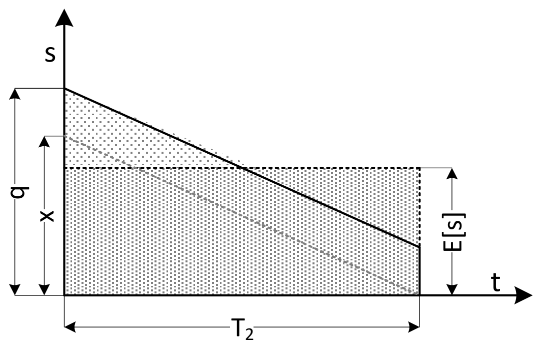

Fig. 1. Average stock when the purchased quantity exceeds demand in T_2

Hence the average stock $\bar{s} = (q - x) + 0.5x$ is a linear function of x. Conversely, as Fig. 2 shows, if $q < x$, the stock linearly decreases until it reaches zero at time $T_e \leq T_2$.

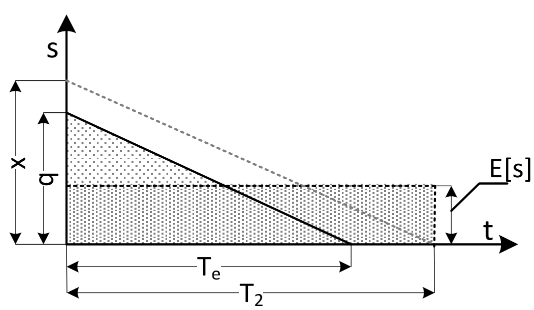

Fig. 2. Average stock when demand in T_2 exceeds the purchased quantity

So, the average stock is $\bar{s} = 0.5q \cdot (T_e/T_2)$ and considering that $T_e = (T_2 q/x)$, we finally have that the average stock $\bar{s} = 0.5 \cdot (q^2/x)$ hyperbolically decreases with respect to x. Owing to these considerations, Eq. (14) can be rewritten as in Eq. (15):

$$\begin{cases} H = hqT_1 + 0.5hT_2\dfrac{q^2}{x} & if\ q \leq x \\ H = hqT_1 + hT_2 \cdot (q - 0.5x) & otherwise \end{cases} \tag{15}$$

To incorporate the disposal and the stockholding cost in the optimization model, their expected value must be computed, as in Eq. (16):

$$E[c\cdot(q-x)^+]+E[h\cdot(qT_1+\bar{s}T_2)]=c\int_0^q(q-x)^+f(x)dx+ \\ +h\cdot\left[qT_1+T_2\cdot\left(\int_0^q(q-0.5x)f(x)dx+\int_q^{+\infty}\frac{q^2}{2x}f(x)dx\right)\right] \tag{16}$$

Limiting the development to the case of a discrete price distribution (but the extension to the continuous case is straightforward), and plugging Eq. (16) into Eq. (11) we finally have:

$$E[\pi]=-qp_0+\sum_{i=1}^{n}\alpha_i p_i\cdot\left(\mu_{x_i}-\int_q^{+\infty}(x-q)^+f_i(x)dx\right)- \\ -\alpha_i\cdot\left(c\int_0^q(q-x)^+f_i(x)dx+h\cdot\left[qT_1+T_2\cdot\left(\int_0^q(q-0.5x)f_i(x)dx+\int_q^{+\infty}\frac{q^2}{2x}f_i(x)dx\right)\right]\right) \tag{17}$$

And the optimal purchase quantity q^* can be easily obtained equating to zero the derivative of Eq. (17) with respect to x.

$$\frac{dE[\pi]}{dq}=-p+\sum\nolimits_{i=1}^{n}\left(\alpha_i-F_i(q)\cdot(\alpha_i p_i+c)-(p+hT_1)-hT_2\cdot\left(F_i(q)+q\int_q^{+\infty}\frac{f_i(x)}{x}dx\right)\right)=0 \tag{18}$$

In fact, Eq. (17) is still a convex function, as Eq. (19) demonstrates:

$$\frac{d^2E[\pi]}{dq^2}=\sum\nolimits_{i=1}^{n}\left(-f_i(q)\cdot(\alpha_i p_i+c)-hT_2\cdot\left(f_i(q)+\int_q^{+\infty}\frac{f_i(x)}{x}dx\right)\right)<0\ \ \forall q \tag{19}$$

4. Optimal solution for a uniformly distributed demand

If demand in T_2 follows an unbounded and continue probability distribution, all the equations presented in the previous sections can be straightforwardly used. For instance, if demand is normally distributed, that is $N(\mu_x,\sigma_x)$, in the simple case of Eq. (5), the optimal quantity can be immediately computed as follows:

$$q^*=\mu_x+\sigma_x\cdot\Phi_{0,1}\left(1-\frac{p}{\bar{p}}\right) \tag{20}$$

where $\Phi_{0,1}(x)$ is the c.d.f. of the standard normal distribution. Unfortunately, using an unbounded distribution is a rather heroic choice: demand in T_2, in fact, must be bounded at least on the left (i.e., $x\geq 0$), and it should be bounded on the right too, since the number of collectors is presumably limited. The use of a bounded distribution, however, requires a little care, as we will show in this section, which considers as an illustrative case the one of a uniformly distributed demand. Note that the use of a uniform distribution is a suitable choice, due to its simplicity and, most of all, due to its high variability, a fact that perfectly matches the type of demand herein considered.

4.1 Single demand uniformly distributed in T_2

When, the price in T_2 can take two values p_h and p_l, with probability α and $(1-\alpha)$, and when demand x is uniformly distributed with $f_l(x)=f_h(x)\sim U[a,b]$, three alternative cases must be considered, as explained next.

- *Case 1.* If q is lower than a, all the purchased quantity q will be certainly sold in T_2 and the profit $\pi_{1,p}$ (for a generic reselling price p) can be computed as in equation (21):

$$\pi_{1,p}=qp-qp_0-hqT_1-h\bar{s}T_2=qp-qp_0-h\cdot\left(qT_1-0{,}5T_2\frac{q^2}{x}\right) \tag{21}$$

- *Case 2.* If q falls within the interval $[a,b]$, the profit $\pi_{2,p}$ can be obtained using the expressions developed in the previous sections, that is:

$$\pi_{2.p} = [x - (x-q)^+] \cdot p - p_0 q - c \cdot (q-x)^+ - h \cdot (qT_1 + \bar{s}T_2) \quad (22)$$

- *Case 3.* If q is higher than b the buyer will certainly have an over stock at the end of T_2. Hence the profit becomes:

$$\begin{aligned}\pi_{3.p} &= x\,p - p_0 q - hqT_1 - c \cdot (q-x) - h\bar{s}T_2 = \\ &= x\,p - p_0 q - c \cdot (q-x) - h \cdot \left(T_1 q + T_2 \cdot (q - 0{,}5x)\right)\end{aligned} \quad (23)$$

So the expected the profit becomes:

$$\begin{cases} E[\pi_1] = q\bar{p} - qp_0 - h \cdot \left(qT_1 - 0{,}5T_2 q^2 \cdot \dfrac{\ln\left(\frac{b}{a}\right)}{(b-a)}\right). \;\; if\ q < a \\ E[\pi_2] = \bar{p}\mu_x - \dfrac{\bar{p} \cdot (b-q)^2}{2 \cdot (b-a)} - p_0 q \dfrac{c \cdot (q-a)^2}{2(b-a)} - \\ \qquad h \cdot \left(qT_1 + T_2 \cdot \left(\dfrac{(q-a) \cdot (3q-a) + 2q^2 \ln\left(\frac{b}{q}\right)}{4(b-a)}\right)\right), \;\; if\ a \le q \le b \\ E[\pi_3] = \bar{p}\mu_x - p_0 q - h \cdot \left(qT_1 + T_2 \cdot (q - 0{,}5\mu_x)\right) - c \cdot (q - \mu_x). \;\; if\ q < b \end{cases} \quad (24)$$

where, as we did before, $\bar{p} = \alpha p_h + (1-\alpha) \cdot p_l$. As it can be seen, the function described by equation (24) is made by an increasing and almost linear part (linear for $h \rightarrow 0$), followed by a convex part and, lastly, by a linearly decreasing part. The optimum value q^*certainly belongs to the interval $[a, b]$ where the function is convex. If holding and disposal costs can be neglected, q^* can be computed as in Eq. (25):

$$q^* = b - \frac{(b-a) \cdot p_0}{\bar{p}} \quad (25)$$

Otherwise, the optimal purchase quantity q^*, in implicit form, is given by Eq. (26):

$$A + B \cdot (q^* - a) - q^* C \ln\left(\frac{b}{q^*}\right) = 0 \quad (26)$$

where:

$$A = \bar{p} - (p_0 + hT_1)$$

$$B = -\frac{(\bar{p} + hT_2 + c)}{(b-a)}$$

$$C = -\frac{hT_2}{(b-a)}$$

4.2 Double demands uniformly distributed in T_2

We now extend the previous case by considering two uniformly distributed demands $f_l(x) \sim U[a_l, b_l]$ and $f_h(x) \sim U[a_h, b_h]$, for price p_l and p_h, respectively. Also, for the sake of simplicity, we will limit the analysis to the pessimistic, but realistic case, of two non-overlapping distributions, with $b_l < a_h$. Let $\pi_{j,k}$ be profit when, relatively to the high-price demand q is in the *j-th* interval with $j \in \{1,2,3\}$, and relatively to the low-price demand q is in the *k-th* interval with $k \in \{1,2,3\}$. We note that the numbers are relative to the position of q with respect to the limits of the distributions, as in cases 1, 2 and 3 of Section 4.1. For instance, when q is smaller than the lower limit of $f_h(x)$ and greater than the upper limit of $f_l(x)$, that is $b_l < q < a_h$, $i = 1$ and $j = 3$ and the profit is denoted as $\pi_{1,3}$. In this case, it is easy to see that the profit is a continuous function defined on five intervals, in each of which $\pi_{j,k}$ is obtained as a linear combination of the profit π_{j,p_h} and π_{k,p_l}, as defined in Eqs. (21-23). For instance, when $q < a$, then $\pi_{1,1} = \left(\alpha\pi_{1,p_h} + (1-\alpha) \cdot \pi_{1,p_l}\right)$, where $\pi_{1,i}$ computed as in equation (21) for $p = p_i$. By extending this reasoning to the other four interval, the expected profit can be finally obtained as in Eq. (27):

$$\begin{cases} E[\pi_{1.1}] = \alpha E[\pi_{1.p_h}] + (1-\alpha) \cdot E[\pi_{1.p_l}]. \;\; if\ q < a_l \\ E[\pi_{1.2}] = \alpha E[\pi_{1.p_h}] + (1-\alpha) \cdot E[\pi_{2.p_l}], \qquad if\ \ a_l \le q < b_l \\ E[\pi_{1.3}] = \alpha E[\pi_{1.p_h}] + (1-\alpha) \cdot E[\pi_{3.p_l}]. \;\; if\ \ b_l \le q < a_h \\ E[\pi_{2.3}] = \alpha E[\pi_{2.p_h}] + (1-\alpha) \cdot E[\pi_{3.p_l}]. \;\; if\ \ a_h \le q < b_h \\ E[\pi_{3.3}] = \alpha E[\pi_{3.p_h}] + (1-\alpha) \cdot E[\pi_{3.p_l}]. \;\; if\ q \ge b_h \end{cases} \quad (27)$$

where, $E[\pi_{j,p_h}]$ and $E[\pi_{k,p_l}]$can be computed as in Eq. (23), by substituting $\bar{p}$ with p_h and with p_l. The function described by Eq. (27) has a trend similar to that of Eq. (24) as it is made by three almost linearly increasing parts (with different slope) followed by a clearly convex part and, lastly by a linearly decreasing part (see also Fig. 3 in the next section as an example). Consequently, if the holding and disposal costs can be neglected, the optimal purchase quantity can be found using Eq. (28).

$$q^* = b_h - \frac{p_0}{\alpha p_h} \cdot (b_h - a_h) \tag{28}$$

Otherwise, the optimal quantity in implicit form is given by Eq. (29).

$$\alpha A + \alpha B \cdot (q^* - a_h) - \alpha q^* C \ln\left(\frac{b_h}{q^*}\right) - (1-\alpha) \cdot D = 0 \tag{29}$$

where:

- A, B and C are computed as before, using p_h instead of $\bar{p}$,
- $D = p_0 + c + h \cdot (T_1 + T_2)$.

To conclude this section, we finally note that Eq. (28) and Eq. (29) hold provided that the optimal quantity falls within the fourth interval on which the curve is defined i.e. if $q^* \in [a_h, b_h]$. This condition is very common, although not always granted. In some extreme cases, for large values of h and low values of the probability alpha, the optimum could be lower than a_h. However, even in this case the optimum value is easy to be found. Indeed, should the optimum obtained either with equation (28) or (29) fall outside of $[a_h, b_h]$, the optimization procedure can be repeated moving leftward, searching a minimum on the immediately precedent interval, until the obtained value does not fall on the considered interval. Since this case is extremely rare, the derivatives needed to compute the optimum in the other intervals, easy to be computed, are not reported.

5. Numerical comparisons

To better visualize how the profit and the optimal quantity q^* change, depending on the value of the α parameter and of the stockholding cost h, a numerical comparison is provided next, based on the following values, with prices in Euros and time units in days:

- $\alpha = \{0.99, 0.95, 0.9, 0.85, 0.8, 0.75, 0.7, 0.65, 0.6\}$,
- $h = \{0.0004, 0.0006, 0.0008, 0.001\}$,
- $p_0 = 3, p_h = 6, p_l = 1$,
- $c = 0.5$,
- $T_1 = 90, T_2 = 360$,
- $f_l(x) \sim U[0,30]$ and $f_h(x) \sim U[60,100]$.

Please note that the range chosen both for α and for h is almost comprehensive. In fact, should alpha be lower than 0.6, the risk incurred by the opportunistic buyer would be definitely too high. Concerning h, a value of 0.0004 is a reasonable lower bound, as this value corresponds to a yearly holding cost per unit of $(0.0004 \cdot 360) = 0.144$, 5% of the original purchasing price p_0. Similarly, $h = 0.001$ is a reasonable upper bound, as the yearly cost per unit is 0.36, 12% of the original purchasing price. As a first example, a graphical comparison (for $h = 0.0004$, and $\alpha = 0.65$) is given in Fig. 3, which shows the four cases discussed in section 4.2. (i.e., one or two demands in T_2 with or without stockholding and disposal costs).

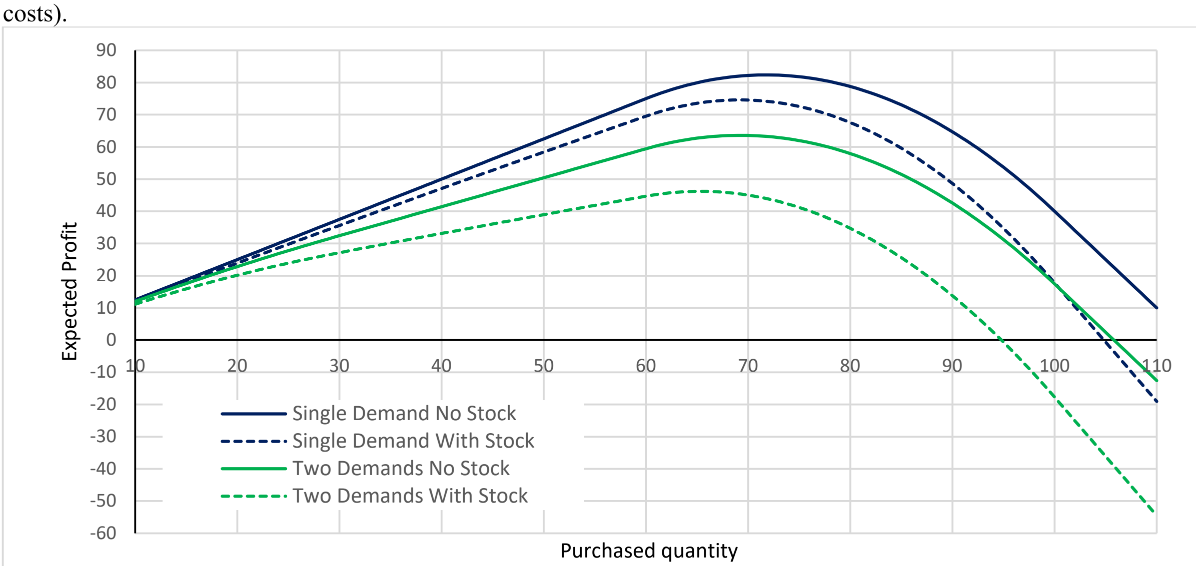

Fig. 3. Comparison of the four different expected profits

As the figure confirms, the effect of the stockholding costs is relevant, especially if two alternative demand profiles can take place in T_2. When demand is low, in fact, the buyer faces a high probability to remain with a residual stock and the expected stockholding and disposal costs get higher. For instance, in the four cases considered in Fig. 3, the optimal purchasing quantities and the corresponding maximal profits are the following ones:

1. Single demand, with no stockholding and disposal cost: $q^* = 72,\ \pi^* = 82.3$,
2. Single demand, with stockholding and disposal cost: $q^* = 69,\ \pi^* = 74.6$,
3. Double demand, with no stockholding and disposal cost: $q^* = 68,\ \pi^* = 63.4$,
4. Double demand, with stockholding and disposal cost: $q^* = 65,\ \pi^* = 45.6$.

As it can be seen, moving from case one to four, the optimal quantity reduces by seven units and the optimal profit drops by 45%. The effect of factors α and h are also confirmed by the more comprehensive analysis provided by Figures 4 and 5, which show the optimal condition at different levels of α and for a null, low (Hl) and high (Hh) value of h. In both figures, data of the single demand models (1D) are in blue, that of the double demand models (2D) are in green.

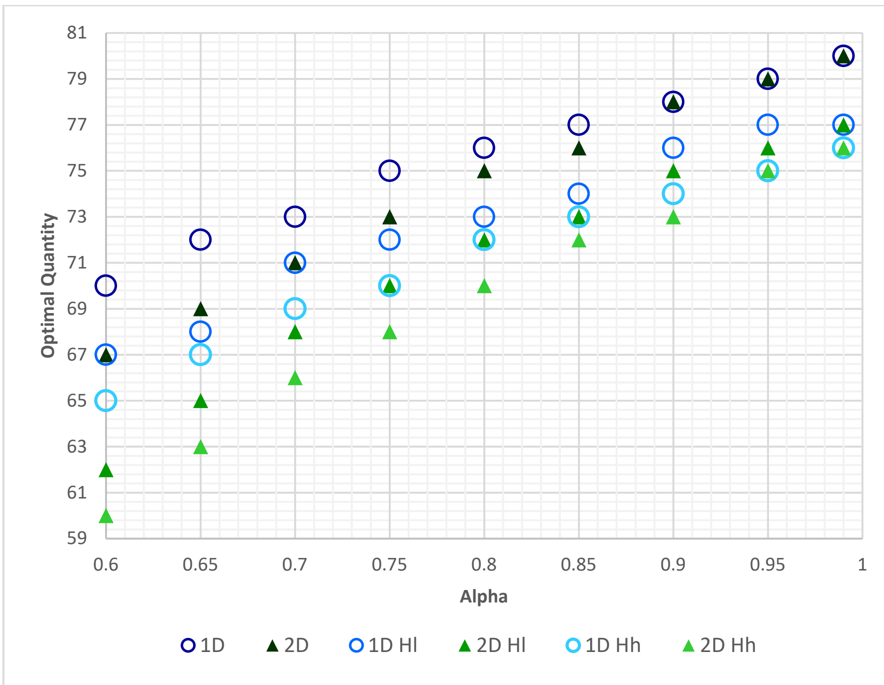

Fig. 4. Optimal purchasing quantity

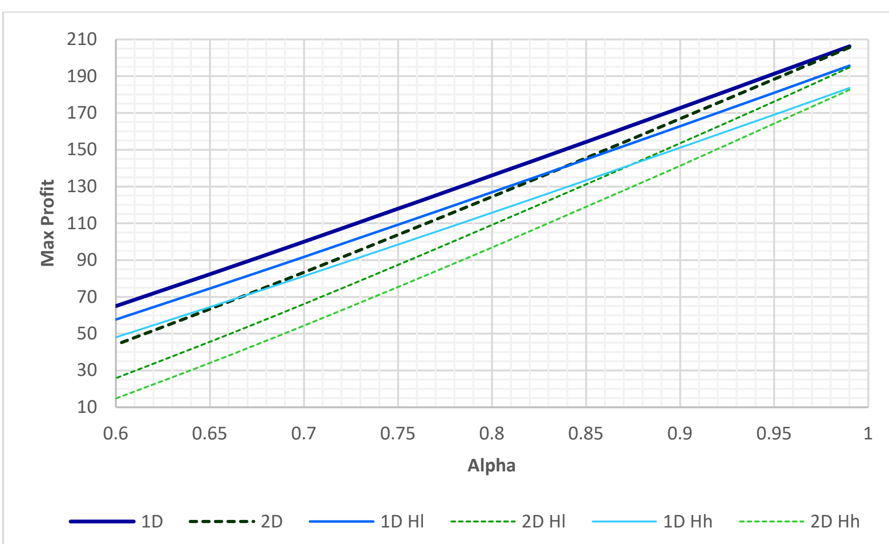

Fig. 5. Maximum expected profit

Specifically, Fig. 4 shows that the gap increases more than linearly as alpha increases. The same results are provided by Fig. 5, which also shows the higher slope of the two-demands profit curves (in green). The effect of α and h, limited to the worst-case scenario (i.e., double demand with stockholding and disposal cost) is finally demonstrated by Fig. 6 and Fig. 7.

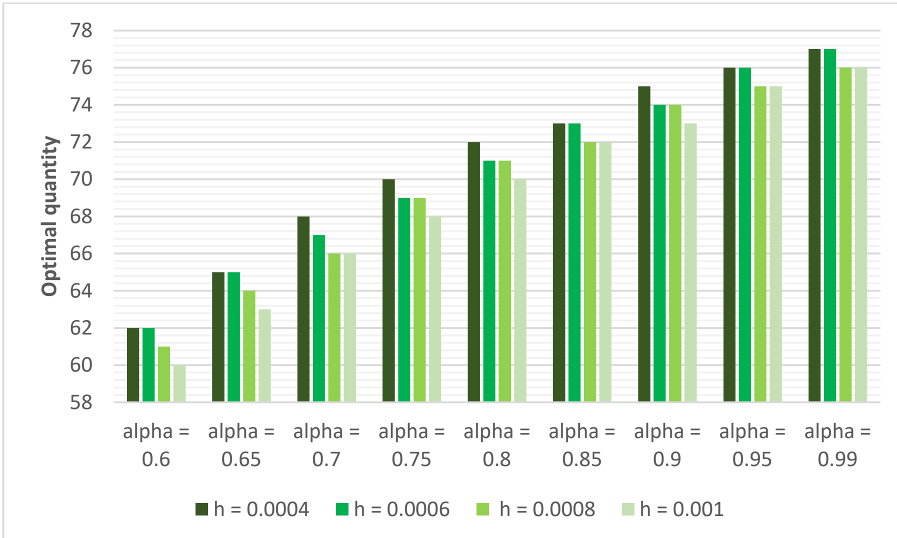

Fig. 6. Optimal quantity, double demand with stockholding costs

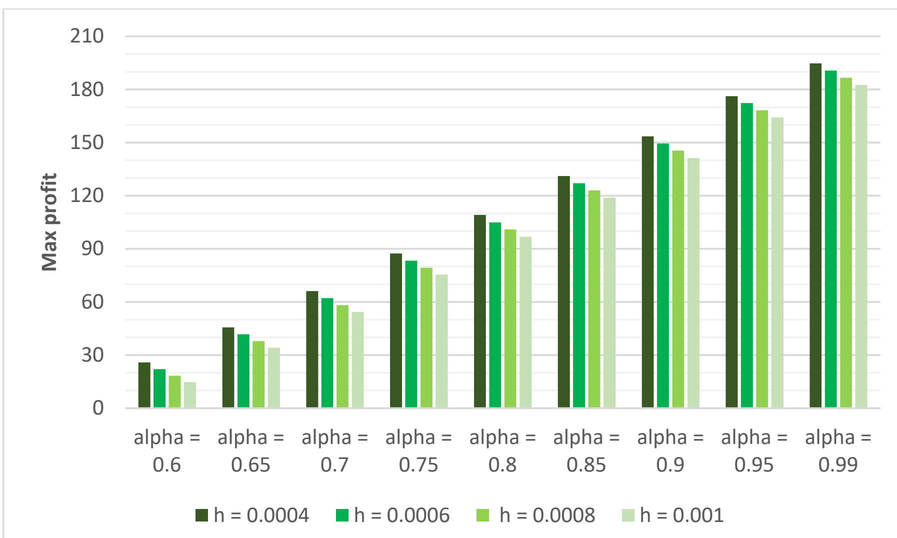

Fig. 7. Maximum profit, double demand with stockholding costs

Clearly, the effect of alpha is very high both in terms of q^* and π^*. Instead, since the stockholding costs are a small part of the total costs, the effect of h is more contained, although increasing as alpha gets higher. Also note that, for high values of alpha, although the expected profit differs, the optimal quantities at different level of h may even coincide.

This does not mean that, in terms of q^*, the effect of h could always be neglected. As shown by Table 1, in fact, should the buyer use the optimal quantity computed neglecting the stockholding costs, namely q_{ns}, the profit loss may be rather significant. For example, in the worst case, when $\alpha = 0.6$ and $h = 0.001$, purchasing 67 units rather than 60, the profit loss of the buyer would be 12.85%.

Table 1
Profit loss neglecting stockholding costs.

Alpha	Holding Cost	q^*_{ns} *no stock*	q^*_{ws} *with stock*	Profit Loss using q^*_{ns} rather than q^*_{ws}
0.9	0.0004	78	75	0.05%
	0.0006	78	74	0.10%
	0.0008	78	74	0.25%
	0.0010	78	73	0.40%
0.8	0.0004	75	72	0.20%
	0.0006	75	71	0.30%
	0.0008	75	71	0.70%
	0.0010	75	70	0.80%
0.7	0.0004	71	68	0.40%
	0.0006	71	67	0.70%
	0.0008	71	66	1.10%
	0.0010	71	66	2.10%
0.6	0.0004	67	62	2.90%
	0.0006	67	62	5.02%
	0.0008	67	61	8.10%
	0.0010	67	60	12.85%

6. Robustness analysis

The only simplifying assumption that we made, concerns the linearity of the stock consumption in period two. Specifically, for the consumption to be linear, the daily demand x_i should be equal in each day i, that is $x_1 = x_2, = \cdots = x_i = \cdots = x_{T_2} = (X/T_2)$, where X is the random variable corresponding to the total demand in T_2. The more X is evenly partitioned in T_2, the more the hypothesis of a linear consumption holds. Unfortunately, this may not always be the case; even if X is a uniformly distributed random variable, rather than being evenly subdivided, in fact, the daily demand could have some spikes and/or it could intensify in a certain interval $\tau \subset T_2$.
To assess the effect the actual partition of X has, both on the optimal quantity and profit, a comprehensive simulation analysis was carried out.

Specifically, five characteristic trends were considered:

- *Smooth (Sm)* – This is the benchmark case, where X is perfectly partitioned over T_2.
- *Smooth with spikes (Ss)* – Also in this case X is evenly partitioned in T_2, but the daily demand can be higher than (X/T_2).
- *Left concentrated demand (Lc)* – Daily demand can have spikes and it is more frequent at the beginning of T_2.
- *Right concentrated demand (Rc)* – As before, but demand is more frequent toward the end of T_2.
- *Centered concentrated demand (Cc)* – As before, but demand is more frequent around the middle of T_2.

A graphical example of the above-mentioned partitions is given by Figure 8, relative to the case $X = 70\ [units]$, $T_2 = 350\ [days]$. Note that since $(X/T_2) = 0.28$, due to the integrity constraint of the daily demand, in the smooth case $x_i \in \{0,1\}\ \forall\ i$. Conversely, in all the other cases a maximum spike of 3 units is allowed. For reasons of space the right concentrated demand has not been shown, this case is indeed perfectly symmetric to the left concentrated one.

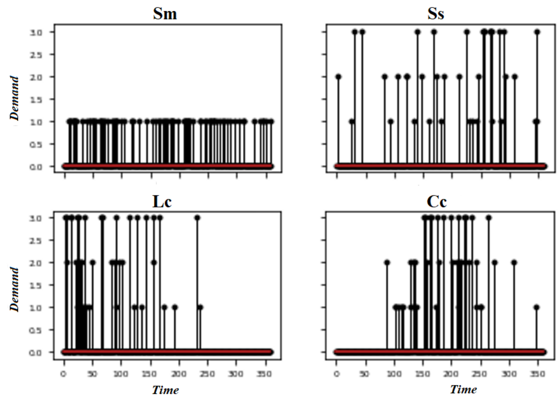

Fig. 8. Possible demand patterns for X = 70, T_2 = 350

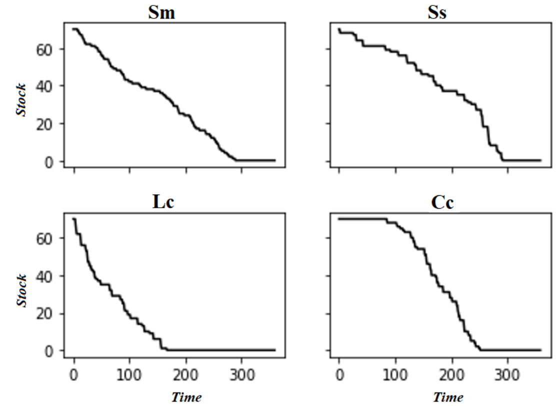

Fig. 9. Possible stock consumptions, for X = 70, q = 80 ad T_2 = 250

For convenience, the stock consumption is also provided by Figure 9, for an initial stock of 80 units.

More precisely, we organized the simulation as follows:

- We considered a uniformly distributed demand $X \sim [60, 100]$ on a period of length $T_2 = 350$ days (the same values used in the numerical examples of subsection 4.3).
- For each possible realizations of the demand (i.e., $X = 60, X = 61, \dots, X = 100$) and for each considered patterns, we randomly generated 1,000 alternative partitioning of the daily demand, for a total of $(41 \cdot 1000) \cdot 5 = 205{,}000$ runs.
- For each simulation run, we generated the stock consumption curves, considering any reasonable value of the purchased quantity $q \in \{60, 61, \dots, 110\}$.
- We finally computed the experimental average of the stock observed in each investigated configuration.

The expected values of the stock are shown, as a function of q, in Figure 10, where the curve labelled as "Ex" corresponds to the expected average stock analytically computed with Eq. (16).

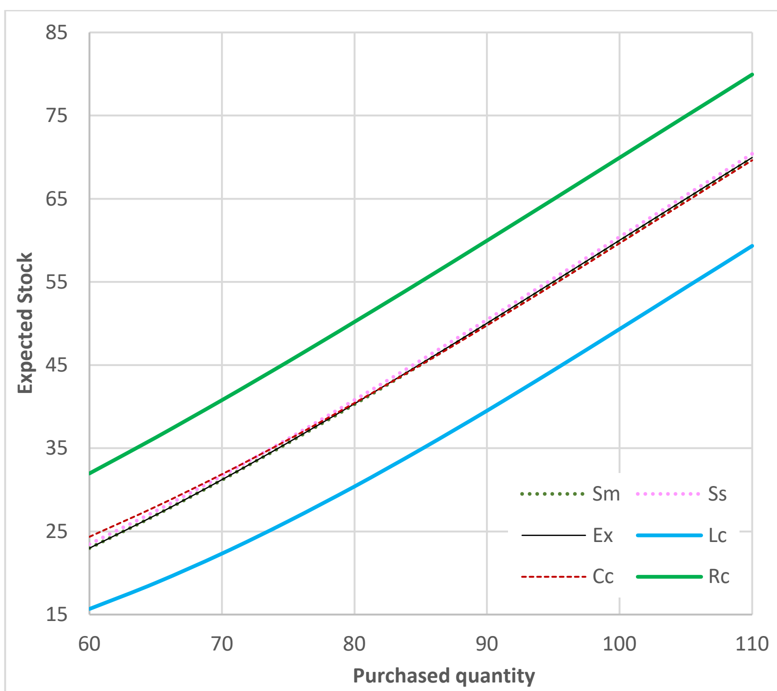

Fig. 10. Simulated stock-consumption trends, for X = 70, q = 80 ad T_2 = 250

As expected, the curves relative to the "smooth" and "smooth with spikes" cases are practically indistinguishable from the "expected" curve, and so the linear approximation is almost perfect. Please note that also a "center concentrated" demand partitioning generates an almost linear stock consumption; this is a particularly relevant result because, among all, the "center concentrated" pattern is probably the most plausible demand pattern.

Conversely, both the "left" and "right" cases deviate rather noticeably from the theoretical curve. In the left case, in fact, the stock decreases much faster than in the linear one, whereas it decreases much slower in the complementary "right" case. However, even though the error seems appreciable, the impact it has in terms of the optimal purchase quantity q^* is negligible, as proved by Table 2, which shows, for each combination of parameters α and h, the optimal quantity q^* and the optimal expected profit π^*, estimated both analytically and by simulation.

Table 2
Comparisons among analytical and simulation results

	Expected		Ss		Lc		Cc		Rc	
$\alpha - h$	q^*	π^*	q^*	π^*	q^*	π^*	q^*	π^*	q^*	π^*
0.9 - 0.004	76	162.792	76	162.039	76	164.722	76	161.852	75	159.071
0.9 - 0.006	75	158.865	75	153.646	75	157.802	75	162.08	75	158.078
0.9 - 0.008	74	154.954	74	148.255	75	153.762	75	159.466	74	154.161
0.9 - 0.001	74	151.117	74	142.908	74	149.781	74	156.878	74	150.285
0.8 - 0.004	73	126.874	73	123.278	73	125.999	73	128.82	73	126.214
0.8 - 0.006	73	123.121	73	118.016	73	122.101	72	126.329	73	122.434
0.8 - 0.008	72	119.416	72	112.867	72	118.263	72	123.846	72	118.742
0.8 - 0.001	71	115.745	71	107.721	72	114.42	72	121.423	71	115.038
0.7 - 0.004	70	91.759	70	88.305	71	90.957	71	93.73	70	91.21
0.7 - 0.006	70	88.226	70	83.31	70	87.264	70	91.409	70	87.672
0.7 - 0.008	69	84.728	69	78.381	70	83.6	69	89.101	69	84.162
0.7 - 0.001	69	81.301	69	73.496	69	79.994	69	86.831	68	80.737
0.6 - 0.004	67	57.708	68	54.458	67	56.999	67	59.693	67	57.321
0.6 - 0.006	67	54.439	66	49.754	67	53.537	67	57.567	66	54.018
0.6 - 0.008	66	51.217	66	45.109	67	50.114	66	55.45	65	50.831
0.6 - 0.001	65	48.035	66	40.534	65	46.726	65	53.366	65	47.646

Specifically, in terms of the expected profit the error is modest (always less than 5%), while the optimal quantities analytically computed coincide, almost perfectly, with those obtained by simulation in each considered configuration. These outcomes clearly demonstrate the validity of our assumption, as well as the robustness of the model developed.

7. Conclusions and future works

In this paper we focused on the NvP, one of the most famous operational research problems in the field of inventory management. Generally, the NvP considers perishable or seasonable items that lose value as time passes; in this paper, instead, we reformulated the NvP, by considering items whose value has an opportunity to rise over time, as for collectible items or the like. We named this new version as the ONvP because, in this case, the buyer should purchase at a standard price a certain quantity q of collectible items, aiming to resell them at a surplus price, immediately after they are discontinued and retired from the market. Specifically, we investigated alternative scenarios, differentiating in terms of demand and prices patterns and of stockholding and/or opportunity cost (of the invested capitals). In each of these cases, we derived analytical formulae to compute the optimal quantity that maximizes the expected profit. Also, a numerical analysis showed that, if demand is limited on a certain range R, the optimal quantity q^* always belong to R and it moves toward the upper extreme of R as the surplus price and the probability alpha (which is the probability that the selling price increases) jointly increase. Conversely, as alpha and the surplus price decrease, q^*get closer to the average value of the demand, and it might even be lower than that, in particularly unfavourable conditions for the buyer. Reduction of q^* and, consequently, of the expected profit $\pi^*$are also due to the stockholding cost H. Although H is generally perceived to be relatively lower that other costs sustained by the buyer, its effect cannot always be neglected: should the buyer use the optimal quantity computed neglecting the stockholding costs, in fact, the profit loss may be rather significant, up to 12% in the worst-case scenario.

Lastly, since our model was developed assuming a linear consumption of the stock (during the selling cycle), a thorough simulation analysis was carried out to assess the validity and robustness of this hypothesis. Obtained outcome confirmed the quality of our approach, as the hypothesis of a linear consumptions holds for the most common demand patterns; even for more extreme cases, the introduced error is almost negligible. We believe that the present work could have interesting implications for managers, dealers and collectors. Firstly, because it extends the NvP to a niche sector that has never been considered by the NvP itself, to the best of our knowledge. In doing so, we derive several analytical formulae that allow to calculate the optimal purchase quantity and the expected profit under different assumptions. Also, and maybe these results are more relevant for practitioners, the analytical formulae were derived in different and realistic demand distributions, and they were proved to be both effective and robust by some numerical analyses.

Specifically, our assumptions respectively neglect and comprehend the stockholding cost. In the first case, we investigated the basic ONvP, whose assumptions were subsequently relaxed by considering a continuous distribution of the reselling price and a reselling price which depends upon the demand in the typically higher-priced period two. In the second case, that is when we considered the stockholding costs, equations were derived for both the cases when the purchased quantity exceeds and does not exceed demand in T_2.

We note that all the approaches, and the relative equations we reported above, can be directly applied in the case of unbounded and continue probability distribution functions. However, due to the fact that unbounded distributions could not fit for the problem at hand, both because demand must only consider non-negative values, and because an upper bound on demand is reasonable, we adapted our approach to the case of bounded demand distributions: namely, we adapted the ONvP to the cases of single and double demands (with different sale prices) uniformly distributed in T_2.

Finally, we reported some numerical analyses to better understand and compare the expected profit, the optimal purchasing quantity and the maximum expected profit under different conditions, that are single and double demand distributions, with or without stockholding costs, and with different values of holding costs and of probability of price increase (α). We also note that a final subsection analyses the robustness of the ONvP, by considering different demand patterns.

To conclude, we note that all expressions were derived in general terms and are valid for any known distribution of the demand. Yet, numerical analyses were limited to the case of a uniformly distributed demand; hence it could be interesting to perform additional tests to see if and how results could change using alternative and more complex demand distributions. This might be the starting point for future research activities. Also, we assumed that the reselling price in T_2 follows a certain discrete probability distribution. However, once a certain price, say p_i, has occurred (i.e., the realization of the price), this value remains constant throughout T_2. To further extend and generalize the model, rather than considering a fixed price p_i, it could be interesting to consider a certain cost function $f_i(t)$ over T_2, such as a linear, quadratic, or exponential trend. This could be another topic for future research activities.

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Appendix A. Notation

For the sake of clarity, the notation used in the paper is provided in the following table.

Table A.1.
Used Notation

Parameters	Notation	Units
Length of period 1	T_1	*days*
Length of period 2	T_2	*days*
Standard Purchasing Price in $\boldsymbol{T_1}$	p_0	*€/item*
Optimistic Price in $\boldsymbol{T_2}$	p_H	*€/item*
Pessimistic Price in $\boldsymbol{T_2}$	p_L	*€/item*
Average Price in $\boldsymbol{T_2}$	$\bar{p}$	*€/item*
Price, if random in $\boldsymbol{T_2}$	y	*€/item*
Profit	π	*€*
Probability of Optimistic Price	α	*[-]*
Purchased quantity	q	*[-]*
Demand in $\boldsymbol{T_2}$	x	*Items per time unit*
Expected Demand in $\boldsymbol{T_2}$	μ_x	*Items per time unit*
Holding Cost	H	*€*
Unit holding cost per unit time	h	*€/(item·time)*
Disposal Cost	C	*€*
Unit disposal cost	c	*€/item*
Average stock in $\boldsymbol{T_2}$	$\bar{s}$	*items*
Probability Distribution	$f(x)$	*[-]*
Cumulative Pr. Distribution	$F(x)$	*[-]*