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Free vibration analysis of a non-uniform cantilever Timoshenko beam with multiple concentrated masses using DQEM

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ABSTRACT

In this paper, a differential quadrature element method (DQEM) is developed for free transverse vibration analysis of a non-uniform cantilever Timoshenko beam with multiple concentrated masses. Governing equations, compatibility and boundary conditions are formulated according to the differential quadrature rules. The compatibility conditions at the position of each concentrated mass are assumed as the continuity in the vertical displacement, rotation and bending moment and discontinuity in the transverse force due to acceleration of the concentrated mass. The effects of number, magnitude and position of the masses on the value of the natural frequencies are investigated. The accuracy, convergence and efficiency of the proposed method are confirmed by comparing the obtained numerical results with the analytical solutions of other researchers. The two main advantages of the proposed method in comparison with the exact solutions available in the literature are: 1) it is less time-consuming and subsequently moreefficient; 2) it is able to analyze the free vibration of the beams whose section varies as an arbitrary function which is difficult or sometimes impossible to solve with analytical methods.

Nomenclature

x	Global spatial coordinate
ξ	Dimensionless global spatial coordinate
$x^{(i)}$	Local spatial coordinate of element *i*
$\zeta^{(i)}$	Dimensionless local spatial coordinate of element *i*
L	Total length of the beam
t	Time
$w(x,t),\ W(x)$	Transverse displacement
$\psi(x,t),\ \Psi(x)$	Rotation due to bending
$W^{(i)}$	Transverse displacement of element *i*
$\Psi^{(i)}$	Rotation due to bending of element *i*

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$v^{(i)}$	Dimensionless transverse displacement of element i
$l^{(i)}$	Dimensionless length of element i
A	Cross sectional area
I	Moment of inertia about the neutral axis
A_0	Values of the cross-section at the clamped edge
I_0	Values of the moment of inertia at the clamped edge
k	Shear correction factor
E	Young's modulus of elasticity
G	Shear modulus
υ	Poisson's Ratio
ρ	Mass density
ω	Angular natural frequency of vibration
λ	Dimensionless natural frequency of vibration
r	Slender ratio
N	Number of grid points
$M^{(i)}$	Bending moment in element i
$V^{(i)}$	Transverse force in element i
α_i	Dimensionless value of the ith concentrated attached mass

1. Introduction

Studying the dynamic characteristics of systems with flexible links or components is an essential research endeavor that can provide successful design of mechanisms, robots, machines, and structures. Extensive investigations have been carried out with regard to the vibration analysis of structures carrying concentrated masses at arbitrary positions. Chen (1963) introduced the mass by the Dirac delta function and analytically solved the problem of a vibrating simply supported beam carrying a concentrated mass at its middle section. Laura et al. (1975) studied the cantilever beam carrying a lumped mass at the top, introducing the mass in the boundary conditions. Laura et al. (1983) used Rayleigh-Ritz method to analyze beams subjected to axial forces and carrying concentrated masses. Gurgoze (1984, 1985) used the normal mode summation technique to determine the fundamental frequency of a cantilever beam carrying masses and rotational springs. Liu et al. (1988) used the Laplace transformation technique to formulate the frequency equation for beams with elastically restrained ends, carrying concentrated masses. De Rosa et al. (1955) investigated dynamic behavior of beams with elastic ends carrying a concentrated mass. Rossit and Laura (2001) presented a solution for vibration analysis of a cantilever beam with a spring mass system attached on the free end. In all above-mentioned studies, authors used Bernoulli-Euler beam theory to model simple structures, which is reliable just for slender beams. In order to increase the accuracy and reliability of studies, especially for the beams with low length-to-thickness ratio, Rao et al. (2006) used coupled displacement field method to study about natural frequencies of a Timoshenko beam with a central point mass. Rossit and Laura (2001) extended their previous research for the Timoshenko beam theory.

Differential quadrature element method (DQEM) is a strong numerical method for analyzing the statics and dynamics of the structures with some discontinuities in loading and material property or geometry. Thus, this method is applied to solve many problems especially in the vibration analysis. Chen (2001, 2002, 2005, 2008) solved different vibration problems using DQEM. E.g., he focused on the vibration of non-uniform shear deformable axisymmetric orthotropic circular plates (Chen, 2001), vibration analysis of non-prismatic shear deformable beams resting on elastic foundations (Chen, 2002), in-plane vibration of curved beam structures (Chen, 2005), and out-of-plane vibration of non-prismatic curved beam structures regarding the effect of shear deformation (Chen, 2008). Malekzadeh et al. (2004) proposed a semi-analytical DQEM for the free vibration analysis of thick plates with the two opposite edges simply supported.

In this paper, DQEM analysis of free transverse vibration of non-uniform cantilever Timoshenko beams with multiple concentrated masses is first presented. Then, the accuracy, convergence, and versatility of the proposed DQEM are confirmed by the exact solutions of the uniform Timoshenko beam presented by other researchers. Finally, the effects of quantity, magnitude and position of the concentrated masses are investigated on the natural frequencies.

2. Differential quadrature method (DQM)

According to the differential quadrature rules, derivatives of a function in $x=x_i$ can be expressed in terms of the value of function in throughout domain as

$$\left.\frac{d^r f}{dx^r}\right|_{x=x_i} = \sum_{j=1}^{N} A_{ij}^{(r)} f_j \,, \tag{1}$$

where N is the number of grid points in the x-direction and $A^{(r)}$ is the weighting coefficient associated with the rth order derivative given by Bert and Malik (1996)

$$A_{ij}^{(1)} = \begin{cases} \dfrac{\prod\limits_{m=1, m\neq i,j}^{N} (x_i - x_m)}{\prod\limits_{m=1, m\neq j}^{N} (x_j - x_m)} \,, & (i,j = 1,2,3,\ldots,N;\, i \neq j) \\ \sum\limits_{m=1, m\neq i}^{N} \dfrac{1}{(x_i - x_m)} \,, & (i = j = 1,2,3,\ldots,N) \end{cases} \qquad A_{ij}^{(r)} = \begin{cases} r\left(A_{ii}^{(r-1)} A_{ij}^{(1)} - \dfrac{A_{ij}^{(r-1)}}{x_i - x_j}\right), & (i,j = 1,2,3,\ldots,N;\, i \neq j) \\ -\sum\limits_{m=1, m\neq i}^{N} A_{im}^{(r)} \,, & (i = j = 1,2,3,\ldots,N) \end{cases} \quad 1 < r \leq (N-1). \tag{2}$$

A well-accepted set of the grid points is the Gauss–Lobatto–Chebyshev points given for interval [0, 1] by

$$\bar{x}_i = \frac{1}{2}\left\{1 - \cos\left[\frac{(i-1)\pi}{(N-1)}\right]\right\}, \quad (i = 1,2,3,\ldots,N). \tag{3}$$

The main advantage of this set is compression of the points in two ends which provides high accuracy in estimation of the value of the derivatives of function in the boundary points.

3. Vibration analysis of non-uniform Timoshenko beam with multiple concentrated masses

3.1. Governing equations

Fig. 1 shows a non-uniform cantilever Timoshenko beam with multiple concentrated masses. The Governing equations of free vibration of the bare Timoshenko beam are written as (Timoshenko et al., 1974).

$$\begin{gathered} \frac{\partial}{\partial x}\left\{kGA(x)\left[\frac{\partial w(x,t)}{\partial x} - \psi(x,t)\right]\right\} - \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \\ \frac{\partial}{\partial x}\left[EI(x)\frac{\partial \psi(x,t)}{\partial x}\right] + kGA(x)\left[\frac{\partial w(x,t)}{\partial x} - \psi(x,t)\right] - \rho I(x)\frac{\partial^2 \psi(x,t)}{\partial t^2} = 0 \end{gathered}, \tag{4}$$

where $w(x, t)$, $\psi(x, t)$, $A(x)$, and $I(x)$ are vertical displacement, rotation due to bending, cross-section area and moment of inertia about the neutral axis, respectively. Also, ρ, E and G are mass density, modulus of elasticity and shear modulus, respectively. k is the shear correction factor which is

introduced to make up the geometry-dependent distribution of the shear stress and depends on the shape of section and the Poisson ratio of the material (Kaneko, 1975).
The displacement $w(x, t)$ and rotation due to bending $\psi(x, t)$ can be assumed as the product of the functions $W(x)$ and $\Psi(x)$ which only depend on the spatial coordinate x and a time dependent harmonic function as

$$w(x,t) = W(x)e^{i\omega t} \quad \psi(x,t) = \Psi(x)e^{i\omega t}; \tag{5}$$

Substituting the Eqs. (5) into the set of Eqs. (4), yields the following set of differential equations:

$$\begin{aligned} &\frac{d^2W(x)}{dx^2} - \frac{d\Psi(x)}{dx} + \frac{1}{A^*(x)}\frac{dA^*(x)}{dx}\left[\frac{dW(x)}{dx} - \Psi(x)\right] + \frac{\rho\omega^2}{kG}W(x) = 0 \\ &\frac{EI_0}{kA_0G}\left[\frac{d^2\Psi(x)}{dx^2} + \frac{1}{I^*(x)}\frac{dI^*(x)}{dx}\frac{d\Psi(x)}{dx}\right] + \frac{A^*(x)}{I^*(x)}\left[\frac{dW(x)}{dx} - \Psi(x)\right] + \frac{\rho I_0\omega^2}{kA_0G}\Psi(x) = 0 \end{aligned}, \tag{6}$$

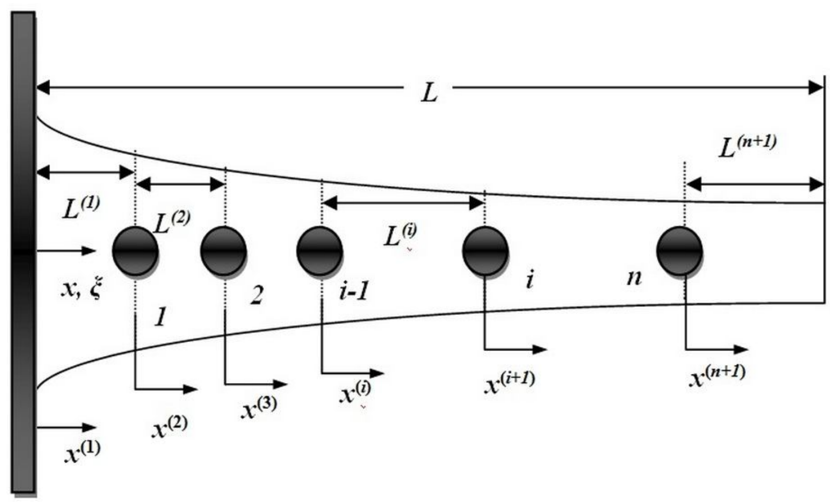

Fig 1. Non-uniform Timoshenko beam with multiple concentrated masses.

where cross-sectional area and the second moment of inertia are written in the following dimensionless form:

$$A^*(x) = \frac{A(x)}{A_0} \quad I^*(x) = \frac{I(x)}{I_0}. \tag{7}$$

where A_0 and I_0 are values of the cross-section and moment of inertia at the clamped edge of the beam, respectively. For the i^{th} sub-beam, the set of Eqs. (6) can be written as

$$\begin{aligned} &\frac{d^2W^{(i)}(x^{(i)})}{d(x^{(i)})^2} - \frac{d\Psi^{(i)}(x^{(i)})}{dx^{(i)}} + \frac{1}{A^*(x)}\frac{dA^*(x)}{dx}\left[\frac{dW^{(i)}(x^{(i)})}{dx^{(i)}} - \Psi^{(i)}(x^{(i)})\right] + \frac{\rho\omega^2}{kG}W^{(i)}(x^{(i)}) = 0 \\ &\frac{EI_0}{kA_0G}\left[\frac{d^2\Psi^{(i)}(x^{(i)})}{d(x^{(i)})^2} + \frac{1}{I^*(x)}\frac{dI^*(x)}{dx}\frac{d\Psi^{(i)}(x^{(i)})}{dx^{(i)}}\right] + \frac{A^*(x)}{I^*(x)}\left[\frac{dW^{(i)}(x^{(i)})}{dx^{(i)}} - \Psi^{(i)}(x^{(i)})\right] + \frac{\rho I_0\omega^2}{kA_0G}\Psi^{(i)}(x^{(i)}) = 0 \end{aligned}. \tag{8}$$

By introducing dimensionless parameters as

$$\xi = \frac{x}{L} \quad \zeta^{(i)} = \frac{x^{(i)}}{L^{(i)}} \quad v^{(i)} = \frac{W^{(i)}}{L} \quad l^{(i)} = \frac{L^{(i)}}{L}, \tag{9}$$

a set of Eqs. (8) can be written as

$$\left(\frac{1}{l^{(i)}}\right)^2 \frac{d^2 v^{(i)}\left(\zeta^{(i)}\right)}{d\left(\zeta^{(i)}\right)^2} - \left(\frac{1}{l^{(i)}}\right)\frac{d\Psi^{(i)}\left(\zeta^{(i)}\right)}{d\zeta^{(i)}} + \frac{1}{A^*(\xi)}\frac{dA^*(\xi)}{d\xi}\left[\left(\frac{1}{l^{(i)}}\right)\frac{dv^{(i)}\left(\zeta^{(i)}\right)}{d\zeta^{(i)}} - \Psi^{(i)}\left(\zeta^{(i)}\right)\right] + \lambda^4 s^2 v^{(i)}\left(\zeta^{(i)}\right) = 0$$
$$s^2\left[\left(\frac{1}{l^{(i)}}\right)^2 \frac{d^2\Psi^{(i)}\left(\zeta^{(i)}\right)}{d\left(\zeta^{(i)}\right)^2} + \frac{1}{I^*(\xi)}\frac{dI^*(\xi)}{d\xi}\left(\frac{1}{l^{(i)}}\right)\frac{d\Psi^{(i)}\left(\zeta^{(i)}\right)}{d\zeta^{(i)}}\right] + \frac{A^*(\xi)}{I^*(\xi)}\left[\left(\frac{1}{l^{(i)}}\right)\frac{dv^{(i)}\left(\zeta^{(i)}\right)}{d\zeta^{(i)}} - \Psi^{(i)}\left(\zeta^{(i)}\right)\right] + \lambda^4 s^2 r^2 \Psi^{(i)}\left(\zeta^{(i)}\right) = 0, \quad (10)$$

where

$$\lambda^4 = \frac{\rho A_0 L^4 \omega^2}{EI_0} \quad s^2 = \frac{EI_0}{kA_0 GL^2} = \frac{2(1+\nu)}{k} r^2 \quad r^2 = \frac{I_0}{A_0 L^2}. \quad (11)$$

By taking the equal grid points for all sub-beams, it can be clearly expressed that

$$\zeta^{(1)} = \zeta^{(2)} = \zeta^{(3)} = \ldots = \zeta^{(i)} = \ldots = \zeta^{(n+1)} = \zeta, \quad (12)$$

Hence, Eqs. (10) can be simplified as

$$\left(\frac{1}{l^{(i)}}\right)^2 \frac{d^2 v^{(i)}(\zeta)}{d\zeta^2} - \left(\frac{1}{l^{(i)}}\right)\frac{d\Psi^{(i)}(\zeta)}{d\zeta} + \frac{1}{A^*(\xi)}\frac{dA^*(\xi)}{d\xi}\left[\left(\frac{1}{l^{(i)}}\right)\frac{dv^{(i)}(\zeta)}{d\zeta} - \Psi^{(i)}(\zeta)\right] + \lambda^4 s^2 v^{(i)}(\zeta) = 0$$
$$s^2\left[\left(\frac{1}{l^{(i)}}\right)^2 \frac{d^2\Psi^{(i)}(\zeta)}{d\zeta^2} + \frac{1}{I^*(\xi)}\frac{dI^*(\xi)}{d\xi}\left(\frac{1}{l^{(i)}}\right)\frac{d\Psi^{(i)}(\zeta)}{d\zeta}\right] + \frac{A^*(\xi)}{I^*(\xi)}\left[\left(\frac{1}{l^{(i)}}\right)\frac{dv^{(i)}(\zeta)}{d\zeta} - \Psi^{(i)}(\zeta)\right] + \lambda^4 s^2 r^2 \Psi^{(i)}(\zeta) = 0. \quad (13)$$

Additionally, to simplify the DQ analogue of the equations, a modified form of the weighting coefficients of element "*i*" is defined as

$$[A]^{(i)} = \frac{[A]^{(1)}}{l^{(i)}} \quad [B]^{(i)} = \frac{[A]^{(2)}}{\left(l^{(i)}\right)^2}. \quad (14)$$

Using the above-mentioned definitions, the DQ analogue of the governing set of equations of element "*i*" becomes

$$\left[B_{ve}\right]^{(i)}\{v\}^{(i)} - \left[A_{se}\right]^{(i)}\{\Psi\}^{(i)} + \lambda^4 s^2\{v\}^{(i)} = 0$$
$$\left[B_{se}\right]^{(i)}\{\Psi\}^{(i)} + \left[A_{ve}\right]^{(i)}\{v\}^{(i)} + \lambda^4 s^2 r^2\{\Psi\}^{(i)} = 0, \quad (15)$$

where

$$\left[B_{ve}\right]^{(i)} = [B]^{(i)} + \left[\frac{1}{A^*(\xi)}\frac{dA^*(\xi)}{d\xi}\right]^{(i)}[A]^{(i)} \qquad \left[A_{se}\right]^{(i)} = [A]^{(i)} + \left[\frac{1}{A^*(\xi)}\frac{dA^*(\xi)}{d\xi}\right]^{(i)}$$
$$\left[B_{se}\right]^{(i)} = s^2\left([B]^{(i)} + \left[\frac{1}{I^*(\xi)}\frac{dI^*(\xi)}{d\xi}\right]^{(i)}[A]^{(i)}\right) - \left[\frac{A^*(\xi)}{I^*(\xi)}\right]^{(i)} \qquad \left[A_{ve}\right]^{(i)} = \left[\frac{A^*(\xi)}{I^*(\xi)}\right]^{(i)}[A]^{(i)}. \quad (16)$$

In the above equation, $[A^*(\xi)/I^*(\xi)]^{(i)}$, $[(dA^*(\xi)/d\xi)/A^*(\xi)]^{(i)}$ and $[(dI^*(\xi)/d\xi)/I^*(\xi)]^{(i)}$ are diagonal geometry-dependent matrices which contain values of the corresponding geometrical parameters. To eliminate the redundant equations, the motion equation should only be written for the domain points (Du et al., 1994; Du et al., 1995; Lin et al., 1994). Therefore, Eqs. (15) should be represented as

$$\left[\overline{B_{ve}}\right]^{(i)}\{v\}^{(i)} - \left[\overline{A_{se}}\right]^{(i)}\{\Psi\}^{(i)} + \lambda^4 s^2\{\overline{v}\}^{(i)} = 0$$
$$\left[\overline{B_{se}}\right]^{(i)}\{\Psi\}^{(i)} + \left[\overline{A_{ve}}\right]^{(i)}\{v\}^{(i)} + \lambda^4 s^2 r^2\{\overline{\Psi}\}^{(i)} = 0, \quad (17)$$

in which, bar signs show the corresponded truncated non-square matrices. By combination of element equations, Eqs. (17) yield

$$\begin{aligned}&[B_v]\{v\}-[A_s]\{\Psi\}+\lambda^4 s^2\{v\}_d=0\\&[B_s]\{\Psi\}+[A_v]\{v\}+\lambda^4 s^2 r^2\{\Psi\}_d=0\text{'}\end{aligned}\tag{18}$$

where

$$\{v\}=\left\{\begin{Bmatrix}v_1^{(1)}\\v_2^{(1)}\\.\\.\\.\\v_{N-1}^{(1)}\\v_N^{(1)}\end{Bmatrix}^T \begin{Bmatrix}v_1^{(2)}\\v_2^{(2)}\\.\\.\\.\\v_{N-1}^{(2)}\\v_N^{(2)}\end{Bmatrix}^T \ \cdots\ \begin{Bmatrix}v_1^{(n+1)}\\v_2^{(n+1)}\\.\\.\\.\\v_{N-1}^{(n+1)}\\v_N^{(n+1)}\end{Bmatrix}^T\right\}^T \quad \{\Psi\}=\left\{\begin{Bmatrix}\Psi_1^{(1)}\\\Psi_2^{(1)}\\.\\.\\.\\\Psi_{N-1}^{(1)}\\\Psi_N^{(1)}\end{Bmatrix}^T \begin{Bmatrix}\Psi_1^{(2)}\\\Psi_2^{(2)}\\.\\.\\.\\\Psi_{N-1}^{(2)}\\\Psi_N^{(2)}\end{Bmatrix}^T \ \cdots\ \begin{Bmatrix}\Psi_1^{(n+1)}\\\Psi_2^{(n+1)}\\.\\.\\.\\\Psi_{N-1}^{(n+1)}\\\Psi_N^{(n+1)}\end{Bmatrix}^T\right\}^T .\tag{19}$$

$$[B_v]=diag\left(\left[\overline{B_{ve}}\right]^{(1)}\ \left[\overline{B_{ve}}\right]^{(2)}\ \cdots\ \left[\overline{B_{ve}}\right]^{(n+1)}\right)\quad [A_s]=diag\left(\left[\overline{A_{se}}\right]^{(1)}\ \left[\overline{A_{se}}\right]^{(2)}\ \cdots\ \left[\overline{A_{se}}\right]^{(n+1)}\right)$$

$$[B_s]=diag\left(\left[\overline{B_{se}}\right]^{(1)}\ \left[\overline{B_{se}}\right]^{(2)}\ \cdots\ \left[\overline{B_{se}}\right]^{(n+1)}\right)\quad [A_v]=diag\left(\left[\overline{A_{ve}}\right]^{(1)}\ \left[\overline{A_{ve}}\right]^{(2)}\ \cdots\ \left[\overline{A_{ve}}\right]^{(n+1)}\right)$$

where "diag" operator provides the diagonal matrices. Eqs. (18) may be rearranged and partitioned in order to separate the boundary, domain, and adjacent displacement and rotation components as (Karami & Malekzadeh, 2002)

$$\begin{aligned}&[B_v]_b\{v\}_b+[B_v]_d\{v\}_d+[B_v]_c\{v\}_c-[A_s]_b\{\Psi\}_b-[A_s]_d\{\Psi\}_d-[A_s]_c\{\Psi\}_c+\lambda^4 s^2\{v\}_d=0\\&[A_v]_b\{v\}_b+[A_v]_d\{v\}_d+[A_v]_c\{v\}_c+[B_s]_b\{\Psi\}_b+[B_s]_d\{\Psi\}_d+[B_s]_c\{\Psi\}_c+\lambda^4 s^2 r^2\{\Psi\}_d=0\text{'}\end{aligned}\tag{20}$$

where

$$\{v\}_b=\begin{Bmatrix}\{v_1^{(1)}\}\\\{v_N^{(n+1)}\}\end{Bmatrix}\quad \{\Psi\}_b=\begin{Bmatrix}\{\Psi_1^{(1)}\}\\\{\Psi_N^{(n+1)}\}\end{Bmatrix}\quad \{v\}_d=\left\{\begin{Bmatrix}v_2^{(1)}\\.\\.\\.\\v_{N-1}^{(1)}\end{Bmatrix}^T \begin{Bmatrix}v_2^{(2)}\\.\\.\\.\\v_{N-1}^{(2)}\end{Bmatrix}^T\ \cdots\ \begin{Bmatrix}v_2^{(n+1)}\\.\\.\\.\\v_{N-1}^{(n+1)}\end{Bmatrix}^T\right\}^T$$

$$\{v\}_c=\left\{\{v_N^{(1)}\}\ \begin{Bmatrix}v_1^{(2)}\\v_N^{(2)}\end{Bmatrix}^T \begin{Bmatrix}v_1^{(3)}\\v_N^{(3)}\end{Bmatrix}^T\ \cdots\ \begin{Bmatrix}v_1^{(n)}\\v_N^{(n)}\end{Bmatrix}^T\ \{v_1^{(n+1)}\}\right\}^T\tag{21}$$

$$\{\Psi\}_c=\left\{\{\Psi_N^{(1)}\}\ \begin{Bmatrix}\Psi_1^{(2)}\\\Psi_N^{(2)}\end{Bmatrix}^T \begin{Bmatrix}\Psi_1^{(3)}\\\Psi_N^{(3)}\end{Bmatrix}^T\ \cdots\ \begin{Bmatrix}\Psi_1^{(n)}\\\Psi_N^{(n)}\end{Bmatrix}^T\ \{\Psi_1^{(n+1)}\}\right\}^T\quad \{\Psi\}_d=\left\{\begin{Bmatrix}\Psi_2^{(1)}\\.\\.\\.\\\Psi_{N-1}^{(1)}\end{Bmatrix}^T \begin{Bmatrix}\Psi_2^{(2)}\\.\\.\\.\\\Psi_{N-1}^{(2)}\end{Bmatrix}^T\ \cdots\ \begin{Bmatrix}\Psi_2^{(n+1)}\\.\\.\\.\\\Psi_{N-1}^{(n+1)}\end{Bmatrix}^T\right\}^T .$$

3.2. Compatibility conditions

In the vicinity of the each concentrated mass $[x_m^-, x_m^+]$, by neglecting the moment of inertia of the concentrated mass, compatibility conditions are continuous in the vertical displacement and rotation due to the bending moment and also discontinuous in the transverse force due to the acceleration of the concentrated mass as

$$w\left(x_m^-,t\right)=w\left(x_m^+,t\right)\quad \psi\left(x_m^-,t\right)=\psi\left(x_m^+,t\right)\quad M\left(x_m^-,t\right)=M\left(x_m^+,t\right)\quad V\left(x_m^-,t\right)-V\left(x_m^+,t\right)=m_i\frac{\partial^2 w\left(x,t\right)}{\partial t^2},\tag{22}$$

where m_i is the translational inertia of the i^{th} concentrated mass and M and V are respectively the bending moment and shear force which are presented for i^{th} sub-beam as (Timoshenko et al., 1974)

$$M^{(i)}=EI\frac{d\Psi^{(i)}}{dx^{(i)}}=\frac{EI}{L}\frac{1}{l^{(i)}}\frac{d\Psi^{(i)}}{d\zeta}\quad V^{(i)}=kAG\left(\Psi^{(i)}-\frac{dW^{(i)}}{dx^{(i)}}\right)=kAG\left(\Psi^{(i)}-\frac{1}{l^{(i)}}\frac{dv^{(i)}}{d\zeta}\right).\tag{23}$$

The compatibility conditions can be indicated in the DQ form as

$$\begin{gathered} v_N^{(i)} = v_1^{(i+1)} \\ \Psi_N^{(i)} = \Psi_1^{(i+1)} \\ \sum_{j=1}^{N} A_{1j}^{(i+1)} \Psi_j^{(i+1)} - \sum_{j=1}^{N} A_{Nj}^{(i)} \Psi_j^{(i)} = 0 \\ \sum_{j=1}^{N} A_{1j}^{(i+1)} v_j^{(i+1)} - \sum_{j=1}^{N} A_{Nj}^{(i)} v_j^{(i)} + \frac{\alpha_i s^2 \lambda^4}{A^*(\xi_i)} v_N^{(i)} = 0 \end{gathered} , \tag{24}$$

where dimensionless translational inertias of the i^{th} concentrated mass is defined as

$$\alpha_i = \frac{m_i}{\rho A_0 L}. \tag{25}$$

Eq. (24) can be rewritten in matrix form as

$$\begin{gathered} [Q_e]^{(i)} \begin{Bmatrix} \{v\}^{(i)} \\ \{v\}^{(i+1)} \end{Bmatrix} + \lambda^4 [q_e]^{(i)} \begin{Bmatrix} \{v\}^{(i)} \\ \{v\}^{(i+1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ [Q_e]^{(i)} \begin{Bmatrix} \{\Psi\}^{(i)} \\ \{\Psi\}^{(i+1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \end{gathered} , \tag{28}$$

where

$$\begin{gathered} [Q_e]_{jk}^{(i)} = \begin{cases} -\delta_{Nk} & j = 1, 1 \le k \le N \\ \delta_{(N+1)k} & j = 1, N+1 \le k \le 2N \\ -A_{Nk}^{(i)} & j = 2, 1 \le k \le N \\ A_{1k}^{(i+1)} & j = 2, N+1 \le k \le 2N \end{cases} \quad \begin{matrix} j = 1,2 \\ 1 \le k \le 2N \end{matrix} . \\ [q_e]_{jk}^{(i)} = \begin{cases} \dfrac{\alpha_i s^2}{A^*(\xi_i)} & j = 2, k = N \\ 0 & else \end{cases} \end{gathered} \tag{29}$$

By writing and composing Eq. (28) for all sub-beams, it can be represented as

$$\begin{gathered} [Q]\{v\} + \lambda^4 [q]\{v\} = \{0\} \\ [Q]\{\Psi\} = \{0\} \end{gathered} \tag{30}$$

where $[Q]$ is the corresponding matrix contains $[Q_e]^{(1)}$ to $[Q_e]^{(n+1)}$ and also $[q]$ is the corresponding matrix contains $[q_e]^{(1)}$ to $[q_e]^{(n+1)}$. Eq. (28) may be rearranged and partitioned in order to separate the boundary, domain, and adjacent displacement components as

$$\begin{gathered} [Q]_b\{v\}_b + [Q]_d\{v\}_d + [Q]_c\{v\}_c + \lambda^4\left([q]_b\{v\}_b + [q]_d\{v\}_d + [q]_c\{v\}_c\right) = \{0\} \\ [Q_v]_b\{\Psi\}_b + [Q_v]_d\{\Psi\}_d + [Q_v]_c\{\Psi\}_c = \{0\} \end{gathered} \tag{31}$$

From Eq. (29), it cab be concluded that $[q]_b$=$[q]_d$=0. Therefore, Eq. (31) can be summarized to

$$[Q]_b\{v\}_b + [Q]_d\{v\}_d + [Q]_c\{v\}_c + \lambda^4 [q]_c\{v\}_c = \{0\} \tag{32-1}$$

$$\{\Psi\}_c = [J_b]\{\Psi\}_b + [J_d]\{\Psi\}_d , \tag{32-2}$$

where

$$[J_b] = -[Q_v]_c^{-1}[Q_v]_b \quad [J_d] = -[Q_v]_c^{-1}[Q_v]_d . \tag{33}$$

Substituting Eq. (32-2) into Eq. (20) leads to

$$\begin{gathered} [B_v]_b\{v\}_b + [B_v]_d\{v\}_d + [B_v]_c\{v\}_c + [G_{sb}]\{\Psi\}_b + [G_{sd}]\{\Psi\}_d + \lambda^4 s^2\{v\}_d = \{0\} \\ [A_v]_b\{v\}_b + [A_v]_d\{v\}_d + [A_v]_c\{v\}_c + [E_{sb}]\{\Psi\}_b + [E_{sd}]\{\Psi\}_d + \lambda^4 s^2 r^2\{\Psi\}_d = \{0\} \end{gathered} \tag{34}$$

where

(35)

$$[G_{sb}]=[A_s]_b+[A_s]_c[J_b] \quad [G_{sd}]=[A_s]_d+[A_s]_c[J_d]$$
$$[E_{sb}]=[B_s]_b+[B_s]_c[J_b] \quad [E_{sd}]=[B_s]_d+[B_s]_c[J_d]$$

3.3. Boundary conditions

The boundary conditions for a cantilever beam depicted in Fig. 1 can be considered as

$$W^{(1)}\Big|_{x^{(1)}=0}=0 \quad \psi^{(1)}\Big|_{x^{(1)}=0}=0 \quad V^{(n+1)}\Big|_{x^{(n+1)}=L^{(n+1)}}=0 \quad M^{(n+1)}\Big|_{x^{(n+1)}=L^{(n+1)}}=0. \tag{36}$$

Eq. (36) can be rewritten using Eq. (23) as

$$\begin{gathered}\Psi^{(1)}\Big|_{x^{(1)}=0}=0 \quad \frac{1}{l^{(i)}}\frac{d\Psi^{(n+1)}}{d\zeta}\Bigg|_{\zeta=1}=0 \\ v^{(1)}\Big|_{x^{(1)}=0}=0 \quad \left(\frac{1}{l^{(i)}}\frac{dv^{(n+1)}}{d\zeta}-\Psi^{(n+1)}\right)\Bigg|_{\zeta=1}=0\end{gathered}. \tag{37}$$

In the DQ form, Eq. (37) can be indicated as

$$[m]\{\Psi\}=0 \tag{38-1}$$
$$[m]\{v\}+[\mathrm{n}]\{\Psi\}=\{0\}, \tag{38-2}$$

where

$$\begin{gathered}\mathrm{m}_{jk}=\begin{cases}1 & j=k=1\\ A^{(n+1)}_{N(k-nN)} & j=2, nN+1\le k\le(n+1)N\\ 0 & else\end{cases} \qquad \begin{matrix} j=1,2 \\ 1\le k\le(n+1)N\end{matrix}. \\ n_{jk}=\begin{cases}-1 & j=2, k=(n+1)N\\ 0 & else\end{cases}\end{gathered} \tag{39}$$

Eqs. (38) may be rearranged and partitioned in order to separate the boundary, domain and adjacent displacement and rotation components as

$$\begin{gathered}[m]_b\{\Psi\}_b+[m]_d\{\Psi\}_d+[m]_c\{\Psi\}_c=\{0\} \\ [m]_b\{v\}_b+[m]_d\{v\}_d+[m]_c\{v\}_c+[\mathrm{n}]_b\{\Psi\}_b+[\mathrm{n}]_d\{\Psi\}_d+[\mathrm{n}]_c\{\Psi\}_c=\{0\}\end{gathered}. \tag{40}$$

From Eq. (39), it can be found that $[n]_c$=$[n]_d$=0; Therefore, using Eq. (32-2), Eq. (40) can be written as

$$\begin{gathered}\{\Psi\}_b=[t]\{\Psi\}_d \\ \{v\}_b=-[m]_b^{-1}[m]_d\{v\}_d-[m]_b^{-1}[m]_c\{v\}_c-[m]_b^{-1}[\mathrm{n}]_b[t]\{\Psi\}_d\end{gathered}, \tag{41}$$

where

$$[t]=-[r]_b^{-1}[r]_d \quad [r]_b=[m]_b+[m]_c[J_b] \quad [r]_d=[m]_d+[m]_c[J_d]. \tag{42}$$

Replacing Eq. (41) into the set of Eqs. (20) and (32-1), new set of equations will be obtained as

(43)

$$[K]\begin{Bmatrix}\{v\}_d\\ \{v\}_c\\ \{\Psi\}_d\end{Bmatrix}=\lambda^4[M]\begin{Bmatrix}\{v\}_d\\ \{v\}_c\\ \{\Psi\}_d\end{Bmatrix},$$

where

$$[K]=\begin{bmatrix}[B_v]_d-[B_v]_b[m]_b^{-1}[m]_d & [B_v]_c-[B_v]_b[m]_b^{-1}[m]_c & [G_{sd}]+[G_{sb}][t]-[B_v]_b[m]_b^{-1}[\mathrm{n}]_b[t]\\ [Q]_d-[Q]_b[m]_b^{-1}[m]_d & [Q]_c-[Q]_b[m]_b^{-1}[m]_c & -[Q]_b[m]_b^{-1}[\mathrm{n}]_b[t]\\ [A_v]_d-[A_v]_b[m]_b^{-1}[m]_d & [A_v]_c-[A_v]_b[m]_b^{-1}[m]_c & [E_{sd}]+[E_{sb}][t]-[A_v]_b[m]_b^{-1}[\mathrm{n}]_b[t]\end{bmatrix}.$$

$$M=-\begin{bmatrix}s^2I_{(n+1)(N-2)*(n+1)(N-2)} & \{0\}_{(n+1)(N-2)*2\mathrm{n}} & \{0\}_{(n+1)(N-2)*(n+1)(N-2)}\\ \{0\}_{2n*(n+1)(N-2)} & [q]_c & \{0\}_{2n*(n+1)(N-2)}\\ \{0\}_{(n+1)(N-2)*(n+1)(N-2)} & \{0\}_{(n+1)(N-2)*2\mathrm{n}} & s^2r^2I_{(n+1)(N-2)*(n+1)(N-2)}\end{bmatrix} \quad (44)$$

Using Eq. (44), natural frequencies and corresponding mode shapes will be determined. Meanwhile, the corresponding mode shapes will be completed by applying the Eqs. (32-2) and (41).
It should be noted that the number of grid points affects on the results. In this paper, the number of grid points should be determinate to satisfy the following relation for convergence of first n frequencies:

$$\left|\frac{\lambda_l^{(N)}-\lambda_l^{(N-1)}}{\lambda_l^{(N-1)}}\right|\le\varepsilon \quad l=1,2,\ldots,n. \quad (45)$$

where ε is considered as 0.01 in this study.

4. Numerical results and discussions

In order to validate the proposed method, a wedge cantilever Timoshenko beam (ν = 0.25, k=2/3) with an attached tip mass (α=0.32) is considered. Table 1 shows the results of solving this problem. The beam cross-sectional properties are assumed to be $A=A_0(1\text{-}0.4\xi)$ and $I=I_0(1\text{-}0.4\xi)^3$. Two different slenderness ratios are considered and the first five non-dimensional natural frequencies are presented. Comparing the results with the exact solutions of Lee and Lin (1995), the excellent accuracy is appreciated.

Table 1. First three non-dimensional frequencies (λ^2) of a wedge cantilever beam with attached mass at the free end

slenderness ratios, r	0.04					0.1				
Mode number	**1**	**2**	**3**	**4**	**5**	**1**	**2**	**3**	**4**	**5**
Presented method	2.117	13.420	36.109	66.697	102.451	1.997	10.695	24.388	40.174	56.739
Lee and Lin (1995)	2.099	13.55	36.76	-	-	2.015	11.07	25.63	-	-

In order to study the effects of the concentrated masses on the natural frequencies, a conical Timoshenko beam (r=0.03, s=0.05) is considered whose diameter varies as $d=d_0(1\text{-}0.5\xi)$ with equally spaced similar concentrated masses (α=1). Table 2 shows the values of the first four dimensionless frequencies for various numbers of concentrated masses. As this table shows, the existence of concentrated masses leads to decrease all the frequencies. Results of Table 2 also show that by increasing the number of the concentrated masses (n), the values of the frequencies decrease. It can be explained by increase of the total mass of the system. As an example, the corresponding modes for n=8 are depicted in Fig. 2.

Table 2. First four dimensionless frequencies of a conical beam with equally spaced similar concentrated masses for various number of concentrated masses

n	λ_1	λ_2	λ_3	λ_4
0	2.141673	4.347607	6.716715	8.972473
1	1.778819	3.400655	6.692156	8.021757
2	1.761166	3.289126	4.798914	7.17954
3	1.32268	2.960147	4.785084	5.586584
5	1.219453	2.706136	4.321166	5.922232
8	0.974544	2.133139	3.38635	4.586028

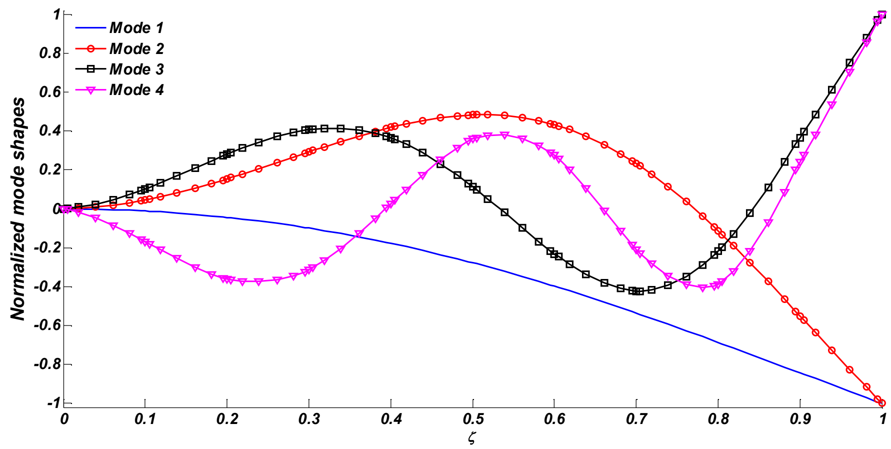

Fig 2. First four modes of a conical beam with eight equally spaced similar concentrated masses.

Magnitude and position of the concentrated masses affect the amount of decrease in the frequencies. E.g., consider a wedge Timoshenko beam (r=0.03, s=0.05) whose width is constant and its height varies as $h=h_0(1\text{-}0.75\xi)$. The first four frequencies of the beam are plotted in Fig. 3 versus the position of a single concentrated mass for different dimensionless values of concentrated masses. From this figure, the values of the frequencies decrease as the value of the mass increases. Also, there are some points in each mode that the value of decrease in frequency is zero when the mass is located on them. These points are nodes in the corresponding mode and the value of the vertical displacement is zero in these points. According to Eq. (22), there is no discontinuity in the transverse force. Also, there are some points that the value of decrease in frequency is maximum when the mass is located on them. These points are antinodes of the corresponding mode shape and the value of the vertical displacement is maximum. It is worth mentioning that the numbers of nodes and antinodes increase at higher modes of vibration.

5. Conclusions

In this study, DQEM analysis of free transverse vibration of a non-uniform cantilever Timoshenko beam was presented. Comparison of the obtained results with the available exact solutions in the literature proved that the accuracy, convergence and subsequently efficiency of the proposed DQEM. The developed DQEM is applicable for the beams with high numbers of spans and able to analyze the non-uniform beams with any variation in the cross section and moment of inertia which are difficult or sometimes impossible to solve by the analytical methods. Numerical investigations on the effect of the numbers, magnitude and position of concentrated masses on the beam frequencies yield the following conclusions:

- Increasing the numbers of concentrated masses, the beam frequencies decrease.
- The values of the frequencies decrease as the magnitude of the masses increases.
- There are some points in each mode that the vertical displacement and the corresponding frequency reduction are zero on them, when mass is located on those points. These points are called nodes.
- There are some points in each mode that the vertical displacement and the corresponding frequency reduction are maximum on them, when mass is located on those points. These points are called antinodes.

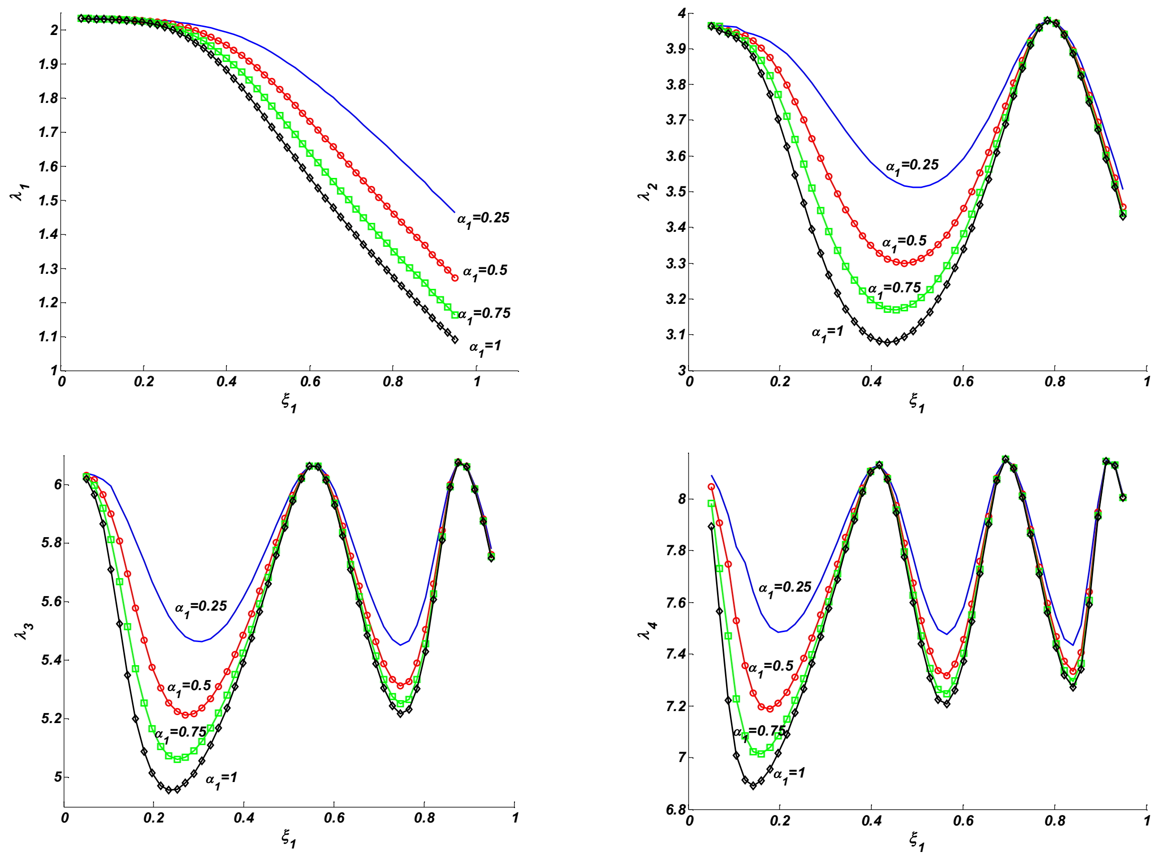

Fig 3. First four dimensionless frequencies of a wedge beam with concentrated masses vs. the position of the mass for various values of the dimensionless magnitude of the mass.

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