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Modeling and analysis of vibration of a gold nano-beam under two-temperature theory

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The present problem deals with the thermo-elastic interaction of a gold nano-beam resonator induced by ramp-type heating under the two temperature theory of generalized thermoelasticity. The governing equations are constructed in the context of two-temperature three-phase-lag model (2T3P) and two-temperature Lord-Shulman (2TLS) model of generalized thermoelasticity. Using the Laplace transform, the fundamental equations have been expressed in the form of a vector-matrix differential equation which is then solved by Eigen value approach and Mathematica software package has been used as a tool. The inversion of Laplace transforms are computed numerically using the method of Fourier series expansion technique. Numerical results for lateral vibration, temperature, displacement, stress, and the strain energy are presented graphically for Lord-Shulman model and also for three-phase lag model. A numerical instance of gold nano-beam in femtoseconds scale has been calculated to present the effect of the ramping time parameter on the entire studied field. The effect of two-temperature parameter is also discussed on the physical fields.

1. Introduction

During the last five decades, non-classical thermoelasticity theories involving hyperbolic type heat transport equations admitting finite speed of thermal signals have been formulated. According to these theories, heat propagation is to be viewed as a wave phenomenon rather than a diffusion phenomenon. Sufficient evidence is available in the literature to show that thermal disturbances do propagate with finite speeds. Experimental investigations conducted on various solids by Ackerman et al. (1967), Ackerman and Guyer (1968) have shown that heat pulses do propagate with finite speeds.

In order to overcome the paradox of the infinite speed of a thermal wave inherent in the classical theory of thermoelasticity (CTE) and classical coupled theory of thermoelasticity (CCTE), efforts were

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made to modify coupled thermoelasticity, on different grounds, to obtain a wave-type heat conduction equation by different researchers. Lord and Shulman (1967) formulated the generalized thermoelasticity theory introducing one relaxation time in Fourier's law of heat conduction and thus transforming the heat conduction equation into a hyperbolic type. Uniqueness of the solution for this theory was proved under different conditions by Ignaczak (1979, 1982), Dhaliwal and Sherief (1980). Green and Lindsay (1972) introduced one more theory, called G-L theory or the temperature-rate-dependent theory (TRDTE), which involves two relaxation times. In this model, Fourier's law of heat conduction is left unchanged, but the classical energy equation and stress-strain temperature relation are modified. On the experimental side, available evidence in support of the existence of finite thermal wave speed in solids is rather sparse, although an experimental study for second sound propagation in dielectric solids and some related experimental observations were reported nearly four decades ago. More detailed discussions on the subject are available in the books of Ignaczak and Ostoja-starzeweski (2009). Ghosh and Kanoria (2009, 2010) have analyzed the thermoelastic response in a functionally graded spherically isotropic hollow sphere. Later, Green-Naghdi (1991) developed three models for generalized thermoelasticity of homogeneous isotropic material which are labeled as models I, II and III. The nature of these theories is such that when the respective theories are linearized, model I reduces to the classical heat conduction theory (based on Fourier's law). The linearized versions of models II and III permit propagation of thermal waves at finite speed. Model II, in particular, exhibits a feature that is not present in the other established thermoelastic models as it does not sustain dissipation of thermal energy (Green and Naghdi; 1992, 1993). In this model, the constitutive equations are derived by starting with the reduced energy equation and by including the thermal displacement gradient among other constitutive variables. The Green-Naghdi's third model admits the dissipation of energy. When Fourier conductivity is dominant, the temperature equation reduces to classical Fourier law of heat conduction, and when the effect of conductivity is negligible, the equation has undamped thermal wave solutions without energy dissipation. Applying the above theories of generalized thermoelasticity, several problems have been solved by Bagri and Eslami (2004, 2007a, 2007b), Kar and Kanoria (2007a, 2007b), Das and Lahiri (2000), Chandrasekharaiah (1996a, 1996b), Ghosh and Kanoria (2008), Islam et al. (2011), etc.

One of these modern theories, the so-called three-phase-lag model, was proposed by Roychoudhuri (2007). According to this model,

$$\vec{q}(P,t+\tau_q) = -\left[K\vec{\nabla}\theta(P,t+\tau_T) + K^*\vec{\nabla}\nu(P,t+\tau_\nu)\right], \tag{1}$$

where $\vec{\nabla}\theta$ is the temperature gradient at a point P of the material at time $t+\tau_T$, $\vec{q}$ is the heat flux vector at the point P of the material at time $t+\tau_q$, $\vec{\nu}\,(\dot{\nu}\equiv\theta)$ is the thermal displacement gradient, K^* is the additional material constant and K is the thermal conductivity of the material.

To study some practical relevant problems and in heat transfer problems involving very short time intervals and in the problems of very high heat fluxes, the hyperbolic equation gives significantly different results than the parabolic equation. According to this phenomenon, the lagging behavior in the heat conduction in solid should not be ignored particularly when the elapsed times during a transient process are very small, say about 10^{-7} second or the heat flux is very much high. Three-phase-lag model is very useful in the problems of nuclear boiling, exothermic catalytic reactions, phonon-electron interactions, phonon-scattering etc., where the delay time τ_q captures the thermal wave behavior (a small scale response in time), the phase-lag τ_T captures the effect of phonon-electron interactions (a microscopic response in space), the other delay time τ_ν is effective since, in the three-phase-lag model, the thermal displacement gradient in considered as a constitutive variable whereas in the conventional thermo-elasticity theory, temperature gradient is considered as a constitutive variable. Several researchers have used three-phase-lag model to solve their problems. Kar and Kanoria (2009) have solved the problem of a functionally graded orthotropic hollow sphere with three-phases-lag effect. The thermo-visco-elastic interaction in an infinite unbounded medium in the presence of a periodically varying heat source under this theory have been solved by Sur and Kanoria (2014a). Sur and Kanoria (2014b) have studied the vibration of a gold nano-beam under this theory. Also, using fractional heat conduction law, several problems under this theory were solved recently (Sur and Kanoria; 2014c, 2014d).

Gurtin and William (1966a, 1966b) have suggested that there are no *a priori* grounds for assuming that the second law of thermodynamics for continuous bodies involve only a single temperature, i.e., it is more logical to assume a second law in which the entropy contribution due to heat conduction is governed by one temperature, that of the heat supply by another. Chen and Gurtin (1968) and Chen et al. (1968, 1969) have formulated a theory of heat conduction in deformable bodies, which depends on two distinct temperatures – the conductive temperature $\phi$and the thermodynamic temperature θ. For time-independent situations, the differences between these two temperatures is proportional to the heat supply, and in the absence of any heat supply, the two temperatures are identical (Chen et al., 1969). For time-dependent problems, however, and for wave propagation problems in particular, the two temperatures are, in general, different, independent of the presence of a heat supply. The key element that sets the two-temperature thermoelasticity (2TT) apart from the classical theory of thermoelasticity (CTE) is the material parameter $a\,(\geq 0)$, called the temperature discrepancy (Chen et al., 1969). Specifically, if a=0, then $\phi=\theta$ and the field equations of the 2TT reduce to those of CTE.

The linearized version of two-temperature theory (2TT) has been studied by many authors. Warren and Chen (1973) have investigated the wave propagation in the two-temperature theory of thermoelasticity. Lesan (1970) has established uniqueness and reciprocity theorems for the 2TT. It should be pointed out that both CTE and 2TT suffer from so-called paradox of heat conduction, i.e., the prediction that a thermal disturbance at some point in a body is felt instantly, but unequally, throughout the body. Although interest in the 2TT has waned since the 1970s, the recent contributions of Quintanilla (2004a, 2004b) and Puri and Jordan (2006) have signaled something of a reversal in this trend. Youssef (2006) has developed theory of two-temperature generalized thermoelasticity based on LS model. El-Karamany et al. (2011) have established uniqueness and reciprocal principles in two-temperature Green-Naghdi thermoelasticity theories. Quintanilla (2008, 2009) has proposed a modification of the 2TT that is based on dual-phase-lag and three-dual-phase-lag heat conduction, respectively.

$$\vec{q}(P,t+\tau_q) = -\left[K\vec{\nabla}\phi(P,t+\tau_T) + K^*\vec{\nabla}\nu(P,t+\tau_\nu)\right], \tag{2}$$

with $\dot{\nu}=\phi$, which is almost Eq. (1) obtained by replacing θ by ϕ.

Youssef and Al-Harby (2007) solved a problem of infinite body with a spherical cavity employing two temperature generalized thermoelasticity by applying the state-space approach. A half space problem filled with an elastic material has been solved in the context of the two-temperature generalized thermoelasticity theory using the state-space approach by Youssef and Al-Lehaibi (2007). Banik and Kanoria (2011, 2012) have studied two-temperature generalized thermoelastic interactions in an infinite body with a spherical cavity. Two temperature generalized thermoelasticity has also been studied by many authors (Sur and Kanoria, 2012, 2014; Youssef, 2008; Islam et al., 2013). Kumar et al. (2010, 2011) have established variational and reciprocal principles and some theorems in two-temperature generalized thermoelasticity.

Many attempts have been made recently to investigate the elastic properties of nanostructured materials by atomistic simulations. Diao et al (2004) studied the effect of free surfaces on the structure and elastic properties of gold nanowires by atomistic simulations. Although, the atomistic simulation is a good way to calculate the elastic constants of nanostructured materials, it is only applicable to homogenous nanostructured materials (e.g., nanoplates, nanobeams, nanowires etc.) with limited number of atoms. Moreover, it is difficult to obtain the elastic properties of the heterogeneous nanostructured materials using atomistic simulations. For these and other reasons, it is prudent to seek a more practical approach. One such approach would be to extend the classical theory of elasticity down to the nanoscale by including in it the hitherto neglected surface or the interface effect. For this, it is necessary first to cast the latter within the framework of continuum elasticity. Nano-mechanical resonators have attracted considerable attention recently due to their many important technological applications. Accurate analysis of various effects on the characteristics of resonators, such as resonant frequencies and quality factors, is crucial for designing high-performance components. Many authors have studied the vibration and heat transfer process of beams. Kidawa (2003) has studied the problem of transverse vibrations of a beam induced by mobile heat source. The analytical solution to the problem was obtained using the Green's function method. Boley (1972) analyzed the vibrations of a simply supported rectangular beam subjected to a suddenly applied heat input distributed along its span. Manolis and Beskos (1980) examined the thermally induced vibration of structures consisting of beams, exposed to rapid surface heating. Al-Huniti et al. (2001) investigated the thermally induced displacements and stresses of a rod using the Laplace transformation technique. Ai Kah Soh et al. (2008a, 2008b) studied the vibration of micro/nanoscale beam resonators induced by ultra-short-pulsed laser by considering the thermoelastic coupling term.

The aim of present contribution is to study the vibration of a gold nano-beam induced by ramp-type laser pulse under the light of two-temperature theory of generalized thermoelasticity based on three-phase-lag model and Lord-Shulman model. Using the Laplace transform, the governing equations have been formulated in Laplace transform domain which is then solved by Eigen value approach. The solutions is space-time domain have been obtained by numerical inversion of Laplace transform which is done by a method of Fourier series expansion technique. Finally the obtained solutions have been depicted graphically to study the effect of ramping parameter. The effect of the temperature discrepancy is also discussed.

2. Formulation of the problem

Since beams with rectangular cross-sections are easy to fabricate, such cross-sections are commonly adopted in the design of NEMS resonators. We consider small flexural deflections of a thin elastic beam of length l $(0 \le x \le l)$, width b $\left(-\frac{b}{2} \le y \le \frac{b}{2}\right)$ and thickness h $\left(-\frac{h}{2} \le z \le \frac{h}{2}\right)$, for which the x, $y$and z axes are defined along the longitudinal, width and thickness directions of the beam, respectively. In equilibrium, the beam is unstrained, unstressed and at temperature θ_0 everywhere.

In the present study, the Euler-Bernoulli assumption (Soh et al., 2008a, 2008b) is adopted, i.e., any plane cross-section, initially perpendicular to the axis of the beam, remains plane and perpendicular to the neutral surface during bending. Thus, the displacement components are given by

$$u_x = -z\frac{\partial w(x,t)}{\partial x}, \quad u_y = 0, \quad u_z(x,y,z,t) = w(x,t). \tag{3}$$

The one-dimensional constitutive equation is

$$\sigma_x = -Ez\frac{\partial^2 w}{\partial x^2} - \beta\theta,$$

where E is the Young's modulus, $\theta = T - T_0$ is the temperature increment, $\beta = \frac{E\alpha_t}{1-2\nu}$, α_t is the coefficient of linear thermal expansion of the material, ν is the Poisson's ratio. Then the flexure moment of the cross-section is given by

$$M(x,t) = -\int_{-\frac{h}{2}}^{\frac{h}{2}} bz\sigma_x dz,$$

i.e., $$M(x,t) = EI\frac{\partial^2 w}{\partial x^2} + b\beta\int_{-\frac{h}{2}}^{\frac{h}{2}} \theta z dz, \tag{4}$$

where $I = \frac{bh^3}{12}$ is the moment of inertia of the cross section. Now, the thermal moment is given by

$$M_T = b\beta \int_{-\frac{h}{2}}^{\frac{h}{2}} \theta z dz.$$

From Eq. (4) we obtain

$$M(x,t) = EI \frac{\partial^2 w}{\partial x^2} + M_T. \tag{5}$$

The equation of transverse motion of a beam is given by

$$\frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} = 0, \tag{6}$$

where ρ is the density, $A = bh$ is the cross-sectional area, EI is the flexural rigidity of the beam. Now, Eq. (5) and Eq. (6) give

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2} + \frac{1}{EI}\frac{\partial^2 M_T}{\partial x^2} = 0. \tag{7}$$

Now, in the context of the three-phase-lag model, the equation of heat conduction takes the following form

$$\left(K^* + \tau_v^* \frac{\partial}{\partial t} + K\tau_T \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_v \ddot{\theta} + \beta\theta_0 \ddot{e}\right), \tag{8}$$

where

$$\phi - \theta = \alpha \nabla^2 \phi, \tag{9}$$

where α (>0) is the two-temperature parameter. $e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$ is the volumetric strain, c_v is the specific heat at constant strain, $K^* = \frac{c_v(\lambda + 2\mu)}{4}$ is a material constant, $\beta = \frac{E\alpha_t}{1-2\nu}$ in which ν is the Poisson's ratio. Also, $\tau_v^* = K + K^*\tau_v$, where τ_v is the phase-lag of the thermal displacement gradient. For $K^* = K$, $\tau_v^* = 0$, $\tau_T = 0$, $\tau_q = \tau_0$ and neglecting τ_q^2, we have 2TLS, model, where τ_0 is the relaxation time for LS model. Now we assume that under two-temperature theory no heat flow across the upper and the lower surface of the beam, so that $\frac{\partial \phi}{\partial z} = 0$ at $z = \pm\frac{h}{2}$. For a very thin beam, we assume that the temperature varies in terms of a $\sin(pz)$ function along the direction of thickness, where $p = \frac{\pi}{h}$. This gives

$$\phi(x,z,t) = \phi_1(x,t)\sin(pz).$$

Hence, Eq. (7) gives

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2} + \left(1 + \alpha p^2\right)\frac{b\beta}{EI}\frac{\partial^2 \phi_1}{\partial x^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\sin(pz)dz - \frac{\alpha b\beta}{EI}\frac{\partial^4 \phi_1}{\partial x^4}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\sin(pz)dz = 0. \tag{10}$$

Again, Eq. (8) reduces to

$$\left[\left(K^* + \tau_v^* \frac{\partial}{\partial t} + K\tau_T \frac{\partial^2}{\partial t^2}\right) + \alpha\rho c_v \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial^2 \phi_1}{\partial x^2} - p^2 \phi_1\right)\sin(pz) = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_v \ddot{\phi}_1 \sin(pz) - \beta\phi_0 z \frac{\partial^2 \ddot{w}}{\partial x^2}\right). \tag{11}$$

After performing the integrations, Eq. (7) reduces to

$$\frac{\partial^4 w}{\partial x^4}+\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}+\left(1+\alpha p^2\right)\frac{2b\beta h^2}{\pi^2 EI}\frac{\partial^2 \phi_1}{\partial x^2}-\frac{2\alpha b\beta h^2}{\pi^2 EI}\frac{\partial^4 \phi_1}{\partial x^4}=0. \tag{12}$$

In Eq. (11), multiplying both sides by z and integrating from $-\frac{h}{2}$ to $\frac{h}{2}$, we get

$$\left[\left(K^*+\tau_v^*\frac{\partial}{\partial t}+K\tau_T\frac{\partial^2}{\partial t^2}\right)+\alpha\rho c_v\frac{\partial^2}{\partial t^2}\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\right]\left(\frac{\partial^2\phi_1}{\partial x^2}-p^2\phi_1\right)=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_v\ddot{\phi}_1-\frac{\beta\phi_0\pi^2 h}{24}\frac{\partial^2\ddot{w}}{\partial x^2}\right). \tag{13}$$

Now, we introduce the following non-dimensional variables

$$x'=\frac{x}{l},\ h'=\frac{h}{l},\ w'=\frac{w'}{l},\ \chi'=\frac{\alpha}{l^2},\ t'=\frac{c_0^2 t}{l},\ \sigma'=\frac{\sigma}{E},\ \phi_1'=\frac{\phi_1}{\phi_0},\ c_0^2=\frac{E}{\rho},$$
$$a_0=\frac{K^*}{\rho c_v c_0^2},\ a_1=\frac{\tau_v^*}{l\rho c_v c_0},\ a_2=\frac{K\tau_T}{l^2\rho c_v},\ b_1=\frac{\tau_q c_0}{l},\ \varepsilon=\frac{\beta}{\rho c_v}.$$

Then, after removing primes, Eq. (12) and Eq. (13) reduces to

$$\frac{\partial^4 w}{\partial x^4}+A_1\frac{\partial^2 w}{\partial t^2}+A_2\frac{\partial^2\phi_1}{\partial x^2}-A_3\frac{\partial^4\phi_1}{\partial x^4}=0 \tag{14}$$

and

$$\left[\left(a_0+a_1\frac{\partial}{\partial t}+a_2\frac{\partial^2}{\partial t^2}\right)+\chi\frac{\partial^2}{\partial t^2}\left(1+b_1\frac{\partial}{\partial t}+\frac{b_1^2}{2}\frac{\partial^2}{\partial t^2}\right)\right]\left(\frac{\partial^2\phi_1}{\partial x^2}-p^2\phi_1\right)=\left(1+b_1\frac{\partial}{\partial t}+\frac{b_1^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2\phi_1}{\partial t^2}-\frac{\varepsilon\pi^2 h}{24}\frac{\partial^2\ddot{w}}{\partial x^2}\right), \tag{15}$$

where

$$A_1=\frac{12l^2}{h^2},\quad A_2=\frac{2b\beta l^3 c_0}{EI\pi^2}\left(1+\frac{\alpha\pi^2}{l^2h^2}\right),\quad A_3=\frac{2b\beta\alpha h^2 l\phi_0}{EI\pi^2}.$$

The medium is initially at rest and the undisturbed state is maintained at a uniform reference temperature. Then we have

$$w(x,0)=\dot{w}(x,0)=\phi_1(x,0)=\dot{\phi}_1(x,0)=0.$$

The problem is to solve the Eq. (14) and Eq. (15) subject to the following boundary conditions

Thermal boundary condition

The plane $x=0$is loaded thermally by ramp-type heating and there is no temperature on the plane $x=l$. Therefore, we have

$$\phi_1(0,t)=\Phi_0\begin{cases}0 & \text{for}\quad t\le 0\\ \frac{t}{t_0} & \text{for}\quad 0<t<t_0\\ 1 & \text{for}\quad t\ge t_0.\end{cases},\qquad \phi_1(l,t)=0. \tag{16}$$

where t_0 is a non-negative constant called the ramp-type parameter and Φ_0 is a constant.

Mechanical boundary condition

We consider that both the ends of the nano-beam are clamped, which gives

$$w(0,t)=\frac{\partial^2 w(0,t)}{\partial t^2}=w(l,t)=\frac{\partial^2 w(l,t)}{\partial t^2}=0. \tag{17}$$

3. Method of solution

Applying the Laplace transform defined by

$$f(s)=\int_0^\infty e^{-st}f(t)\,dt,\quad \mathrm{Re}(s)>0, \tag{18}$$

To Eq. (14) and Eq. (15), we obtain

$$\frac{d^4\bar{w}}{dx^4}+A_1s^2\bar{w}+A_2\frac{d^2\bar{\phi}_1}{dx^2}-A_3\frac{d^4\bar{\phi}_1}{dx^4}=0, \tag{19}$$

and

$$\frac{d^2\bar{\phi}_1}{dx^2}-p^2\bar{\phi}_1=A_3s^2\bar{\phi}_1-A_4s^2\frac{d^2\bar{w}}{dx^2}. \tag{20}$$

where $A_3=\dfrac{1+b_1s+\frac{1}{2}b_1^2s^2}{\left(a_0+a_1s+a_2s^2\right)+\chi\left(1+b_1s+\frac{1}{2}b_1^2s^2\right)}$ and $A_4=\dfrac{\varepsilon\pi^2h}{24}\dfrac{1+b_1s+\frac{1}{2}b_1^2s^2}{\left(a_0+a_1s+a_2s^2\right)+\chi\left(1+b_1s+\frac{1}{2}b_1^2s^2\right)}$.

We consider a new function as follows

$$\frac{d^2\bar{w}}{dx^2}=\bar{U}. \tag{21}$$

Boundary conditions (16) and (17) in the transformed domain take the form

$$\bar{\phi}_1(0,s)=\frac{\Phi_0}{t_0}\left(\frac{1-e^{-t_0s}}{s^2}\right)=G(s),\quad \bar{\phi}_1(l,s)=0, \tag{22}$$

and

$$\bar{w}(0,s)=\bar{U}(0,s)=\bar{w}(l,s)=\bar{U}(l,s)=0. \tag{23}$$

Now, using Eq. (21), Eq. (19) and Eq. (20) is reduced to

$$\frac{d^2\bar{\phi}_1}{dx^2}=\alpha_1\bar{\phi}_1-\alpha_2\bar{U}, \tag{24}$$

$$\frac{d^2\bar{U}}{dx^2}=-\alpha_3\bar{w}+\alpha_4\bar{\phi}_1+\alpha_5\bar{U}, \tag{25}$$

where

$$\alpha_1=\left(p^2+A_3s^2\right),\quad \alpha_2=A_4s^2,\quad \alpha_3=\frac{A_1s^2}{1+\alpha_2A_3},\quad \alpha_4=\frac{\alpha_1(A_3\alpha_1-A_2)}{1+\alpha_2A_3},\quad \alpha_5=\frac{A_2\alpha_2-A_3\alpha_1\alpha_2}{1+\alpha_2A_3}.$$

Choosing the functions $\bar{w}$, $\bar{\phi}_1$, $\bar{U}$, $\frac{d\bar{w}}{dx}=\bar{w}'$, $\frac{d\bar{\phi}_1}{dx}=\bar{\phi}_1'$, $\frac{d\bar{U}}{dx}=\bar{U}'$ in the $x-$ direction, then Eq. (21), Eq. (24) and Eq. (25) can be written in matrix form by using the Bahar-Hetnarski method as follows,

$$\frac{d\bar{V}(x,s)}{dx}=A(s)\bar{V}(x,s), \tag{26}$$

where

$$\bar{V}(x,s)=\left[\bar{w}(x,s)\quad \bar{\phi}_1(x,s)\quad \bar{U}(x,s)\quad \bar{w}'(x,s)\quad \bar{\phi}_1'(x,s)\quad \bar{U}'(x,s)\right]^T \tag{27}$$

and

$$A(s)=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\alpha_2 & 0 & 0 & 0 \\ -\alpha_3 & -\alpha_4 & \alpha_5 & 0 & 0 & 0 \end{bmatrix} \quad (28)$$

The characteristic equation of the matrix $A(s)$ is given by

$$k^6 - lk^4 + mk^2 - n = 0, \quad (29)$$

where

$$l = \alpha_1 + \alpha_5,$$
$$m = \alpha_1\alpha_5 - \alpha_2\alpha_4 + \alpha_3,$$
$$n = \alpha_1\alpha_3.$$

Now, the roots of the characteristic Eq. (29), which are also the Eigen values of the matrix $A(s)$ are of the form

$$k = \pm k_1, \quad k = \pm k_2, \quad k = \pm k_3,$$

where

$$k_1^2 + k_2^2 + k_3^2 = l, \quad (30)$$
$$k_1^2k_2^2 + k_2^2k_3^2 + k_3^2k_1^2 = m, \quad (31)$$
$$k_1^2k_2^2k_3^2 = n. \quad (32)$$

4. Eigen value approach

Consider the vector-matrix differential equation in the following form

$$\frac{d\tilde{v}}{dx} = A\tilde{v}, \quad (33)$$

where

$$\tilde{v} = [v_1, v_2, \ldots, v_n]^T \text{ and } A = (a_{ij});\ i, j = 1, 2, \ldots, n. \quad (34)$$

are real vector and matrix respectively. Let

$$A = V\Lambda V^{-1}, \quad (35)$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

is a diagonal matrix whose elements λ_1, λ_2,…, λ_n are distinct Eigen values of A . Let $\tilde{V}_1$, $\tilde{V}_2$,…, $\tilde{V}_n$ be the Eigen vectors of A corresponding to λ_1 , λ_2 ,…, λ_n respectively, and

$$V = [\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_n] = (x_{ij}) \text{ (say)};\ i, j = 1, 2, \ldots, n. \quad (36)$$

Substituting Eq. (35) in Eq. (33) and pre-multiplying by V^{-1}, we obtain

$$V^{-1}\frac{d\tilde{v}}{dx} = \Lambda V^{-1}\tilde{v},$$

or, $\frac{d}{dx}\left(V^{-1}\tilde{v}\right)=\lambda\left(V^{-1}\tilde{v}\right).$

If we define

$$\tilde{y} = V^{-1}\tilde{v}, \tag{37}$$

We need to solve the equations

$$\frac{d\tilde{y}}{dx}=\Lambda\tilde{y}. \tag{38}$$

This is a set of $n-$ coupled differential equations. Consider the $r-$ th equation, which is typical

$$\frac{d\tilde{y}_r}{dx}=\lambda_r y_r.$$

The solution of the above equation is

$$y_r = C_r e^{\lambda_r x}, \quad r = 1, 2, \dots, n, \tag{39}$$

where C_r are scalars to be determined from the initial conditions.

Since, from Eq. (37), $\tilde{v} = V\ \tilde{y}$, we may write

$$\tilde{v}=\sum_{r=1}^{n}V_r y_r,$$

which can be explicitly written as

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} y_1 + \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} y_2 + \cdots + \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix} y_n. \tag{40}$$

Substituting Eq. (39) in Eq. (40), we get the complete solution of Eq. (33) in the form

$$v_r = c_1 x_{r1} e^{\lambda_1 x} + c_2 x_{r2} e^{\lambda_2 x} + \cdots + c_n x_{rn} e^{\lambda_n x}, \quad r = 1, 2, \dots, n. \tag{41}$$

Now, the Eigen-vector corresponding to the Eigen value k of the matrix $A(s)$ can be written as

$$V = \begin{bmatrix} -\alpha_4 \\ k^4 - k^2\alpha_5 + \alpha_3 \\ -\alpha_4 k^2 \\ -\alpha_4 k \\ k\left(k^4 - k^2\alpha_5 + \alpha_3\right) \\ -\alpha_4 k^3 \end{bmatrix}. \tag{42}$$

Thus, following Eq. (40), we obtain from Eq. (26)

$$\begin{bmatrix} \bar{w}(x,s) \\ \bar{\phi}_1(x,s) \\ \bar{U}(x,s) \\ \bar{w}'(x,s) \\ \bar{\phi}_1'(x,s) \\ \bar{U}'(x,s) \end{bmatrix} = \sum_{r=1}^{6} V_r y_r, \tag{43}$$

where $y_1 = C_1 e^{k_1 x}$, $y_2 = C_2 e^{k_2 x}$,…, $y_6 = C_6 e^{k_6 x}$. Now, substituting the boundary conditions (22) and (23) in the first three equations of Eq. (43) and after some complicated calculations using Mathematica, we obtain the final solutions in Laplace transform domain in the following form as follows:

The lateral deflection is given by

$$\bar{w}(x,s)=\frac{\Delta\sinh\{k_1(l-x)\}}{(k_1^2-k_2^2)(k_1^2-k_3^2)\sinh(k_1 l)}+\frac{\Delta\sinh\{k_2(l-x)\}}{(k_2^2-k_1^2)(k_2^2-k_3^2)\sinh(k_2 l)}+\frac{\Delta\sinh\{k_3(l-x)\}}{(k_3^2-k_1^2)(k_3^2-k_2^2)\sinh(k_3 l)}. \tag{44}$$

The conductive temperature

$$\bar{\phi}(x,z,s)=-\frac{\alpha_2 k_1^2\Delta\sin(pz)\sinh\{k_1(l-x)\}}{(k_1^2-\alpha_1)(k_1^2-k_2^2)(k_1^2-k_3^2)\sinh(k_1 l)}-\frac{\alpha_2 k_2^2\Delta\sin(pz)\sinh\{k_2(l-x)\}}{(k_2^2-\alpha_1)(k_2^2-k_1^2)(k_2^2-k_3^2)\sinh(k_2 l)}-\frac{\alpha_2 k_3^2\Delta\sin(pz)\sinh\{k_3(l-x)\}}{(k_3^2-\alpha_1)(k_3^2-k_1^2)(k_3^2-k_2^2)\sinh(k_3 l)}. \tag{45}$$

The displacement

$$\bar{u}_x(x,z,s)=\frac{z\Delta k_1\cosh\{k_1(l-x)\}}{(k_1^2-k_2^2)(k_1^2-k_3^2)\sinh(k_1 l)}+\frac{z\Delta k_2\cosh\{k_2(l-x)\}}{(k_2^2-k_1^2)(k_2^2-k_3^2)\sinh(k_2 l)}+\frac{z\Delta k_3\cosh\{k_3(l-x)\}}{(k_3^2-k_1^2)(k_3^2-k_2^2)\sinh(k_3 l)}. \tag{46}$$

The strain

$$\bar{e}(x,z,s)=-\frac{z\Delta k_1^2\sinh\{k_1(l-x)\}}{(k_1^2-k_2^2)(k_1^2-k_3^2)\sinh(k_1 l)}-\frac{z\Delta k_2^2\sinh\{k_2(l-x)\}}{(k_2^2-k_1^2)(k_2^2-k_3^2)\sinh(k_2 l)}-\frac{z\Delta k_3^2\sinh\{k_3(l-x)\}}{(k_3^2-k_1^2)(k_3^2-k_2^2)\sinh(k_3 l)}. \tag{47}$$

where

$$\Delta=\frac{G}{\alpha_1\alpha_2}(\alpha_1-k_1^2)(\alpha_1-k_2^2)(\alpha_1-k_3^2).$$

Also, the thermodynamic temperature $\bar{\theta}(x,z,s)$ can be obtained from the non-dimensional form of Eq. (9) by using Eq. (45).

5. The stress and strain energy

According to Hooke's law, the stress along $x-$ axis is given by

$$\sigma(x,z,t)=E(e-\alpha_t\theta). \tag{48}$$

Using the non-dimensional variables, we obtain the stress in the form

$$\sigma_{xx}(x,z,t)=e-\alpha_t\theta_0\theta. \tag{49}$$

Applying Laplace transform, to Eq. (49), we obtain

$$\bar{\sigma}(x,z,s)=\bar{e}-\alpha_t\theta_0\bar{\theta}. \tag{50}$$

The strain energy, which is generated on the beam, is given by

$$W(x,z,t)=\frac{1}{2}\sum_{i,j=1}^{n}\sigma_{ij}e_{ij}=\frac{1}{2}\sigma e=-\frac{1}{2}z\sigma U \tag{51}$$

We can write Eq. (51) as

$$W(x,z,t)=-\frac{1}{2}z\left[L^{-1}(\bar{\sigma}_{xx})\right]\left[L^{-1}(\bar{U})\right], \tag{52}$$

where

$$L^{-1}\left[\bar{f}(s)\right]=f(t).$$

Eqs. (44)-(47), Eq. (50) and Eq. (52) together constitute the complete solutions of the problem in the Laplace transform domain.

6. Numerical results and discussions

To obtain the solution for the lateral deflection, conductive temperature, thermodynamic temperature, displacement, thermal stress and the strain in the space-time domain, we have to apply inversion of the Laplace transform. This has been done numerically using a method based on the Fourier series expansion technique (Honig & Hirdes, 1984). For the purpose of illustration, here we have used the gold (Au) material following Youssef and Elsibai (2011) and Kar and Kanoria (2009). The material constants are given below

$\lambda = 198$ GPa, $\mu = 27$ GPa, $\alpha_T = 14.2\times10^{-6}\,\mathrm{K}^{-1}$, $\rho = 1930$ kg/m^3, $\phi_0 = 293$ K, $c_v = 130$ J kg^{-1} K^{-1}, $E = 180$ GPa, $\nu = 0.44$, $\tau_q = 2.0\times10^{-7}$ s; $\tau_T = 1.5\times10^{-7}$ s; $\tau_v = 1.0\times10^{-7}$ s; $K = 200$, $K^* = 7$,

Which satisfies the stability condition given by Quintanilla and Racke (2004a) that, under three-phase-lag heat conduction that $K^*\tau_q < \tau_v^* < \frac{2K\tau_T}{\tau_q}$. The aspect ratios of the beam are fixed as $\frac{l}{h} = 10$ and $\frac{b}{h} = \frac{1}{2}$. When h varies, l and $b$change accordingly with h.

For the nano-scale beam, we will take the range of the original beam length $l(1-100)\times10^{-12}$ m. The original time t and the ramping-time parameter t_0 will be considered in femtoseconds scale $t(1-100)\times10^{-15}$ sec. The computations are carried out for the wide range of the beam length when $l = 1.0$, $\Phi_0 = 1.0$, $z = h/6$ and $t = 50$.

In order to study the effect of the two temperature parameter, we now present the thermophysical quantities in their graphical representations (Figs. 1-5) for ramping parameter $t_0 = 75$. Fig 1 depicts the variation of the lateral deflection of the beam when $t_0 = 75$ for both one temperature $(\omega = 0.0)$ and two-temperature $(\omega = 0.1)$ theories. From the figure, it is seen that the defection vanishes on both the inner and outer boundaries of the beam which satisfies our theoretical boundary condition. The oscillatory behavior in the propagation of the deflection is observed near the outer boundary of the beam. The magnitude of the deflection is maximum near $x = 0.8$ for one temperature Lord Shulman model. For 3P lag model, the magnitude of oscillation of w is larger for two temperature theory than that of one temperature theory near the outer boundary of the beam.

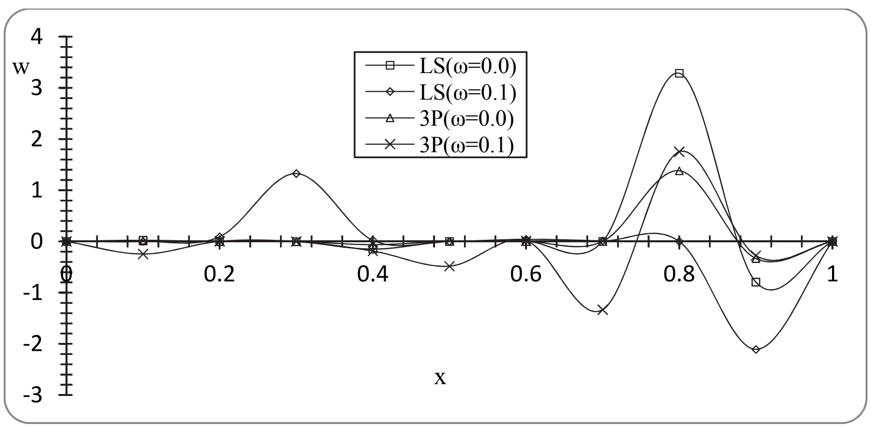

Fig. 1. Variation of w versus x for $\omega = 0.0, 0.1$ and $t_0 = 75$

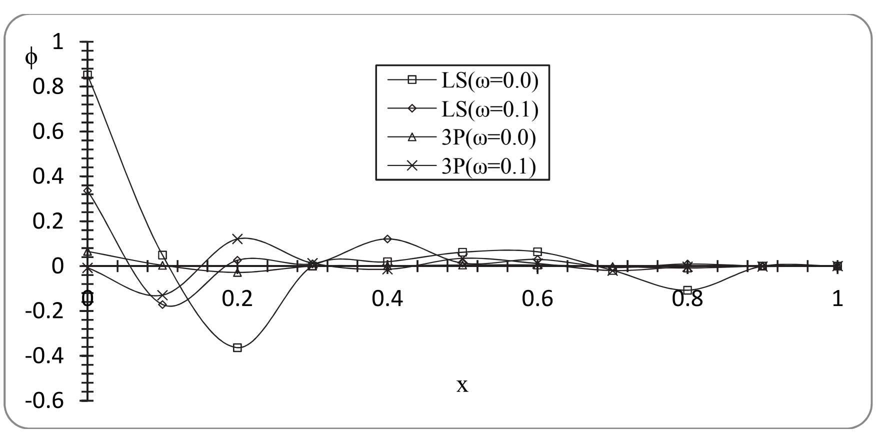

Fig. 2. Variation of ϕ versus x for $\omega = 0.0, 0.1$ and $t_0 = 75$

Fig. 2 is plotted to study the effect of two-temperature parameter on the conductive temperature (ϕ) versus the distance x. As seen from the figure, ϕ attains the maximum magnitude near the inner boundary of the beam. Also, the oscillatory behavior in the propagation is seen near the inner boundary and then it diminishes to zero on the outer boundary which validates the correctness of the thermal boundary condition. For two temperature theory, ϕ almost disappears after $x=0.3$ in 3P lag model, whereas for LS model the effect of the conductive temperature is found for 0 < x < 0.7.

Fig. 3 depicts the variation of the thermodynamic temperature (θ) versus x for $t_0 = 75$ for two-types of temperatures. It is observed that θ is maximum in magnitude near the inner boundary of the beam for two temperature theory $(\omega = 0.1)$ than that of one temperature theory $(\omega = 0.0)$ for both the models. For LS model, θ almost disappears in $0.2<x<1$. Also, for 3P lag model, oscillatory nature in the propagation of the thermodynamic temperature is found.

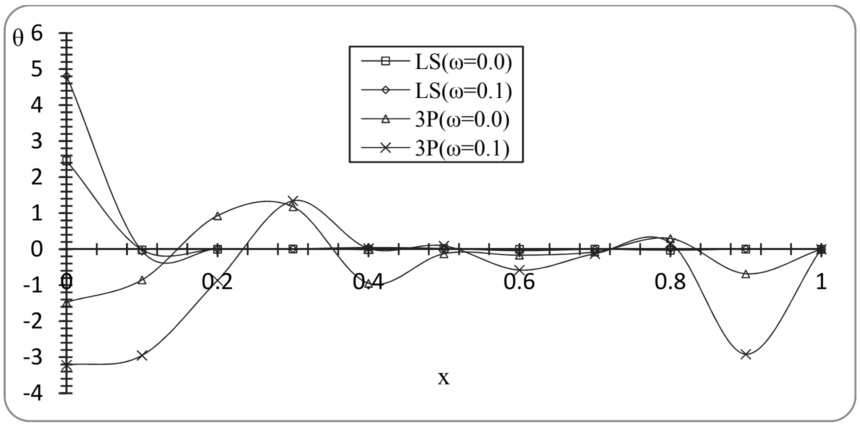

Fig. 3. Variation of θ versus x for $\omega = 0.0, 0.1$ and $t_0 = 75$

Fig. 4 is plotted to show the propagation of the displacement u against the distance x for both one temperature and two temperature theories. It is observed that the displacement component is maximum in magnitude for 3P lag model when $\omega = 0.1$. An oscillatory nature in the propagation of the thermal displacement is observed.

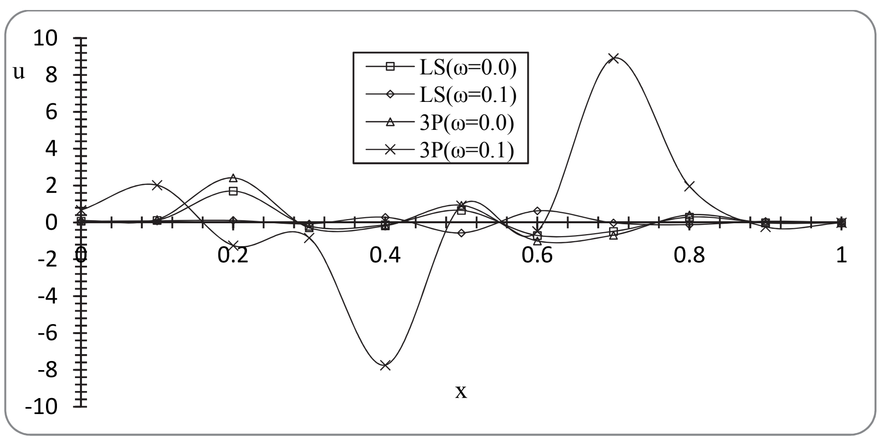

Fig. 4. Variation of u versus x for $\omega = 0.0, 0.1$ and $t_0 = 75$

Fig. 5 depicts the propagation of the elongation versus x for $t_0 = 75$ and for one temperature and two temperature theories. As seen from the figure, there is almost no elongation in the beam in $0 < x < 0.4$ for LS model. It is observed that the elongation is maximum in magnitude for two-temperature $(\omega = 0.1)$ 3P lag model near $x=0.8$. Fig. 6 and Fig. 7 are plotted to study the effect of the ramp-type parameter on the conductive temperature (ϕ) and the thermodynamic temperature (θ) for two-temperature $(\omega = 0.1)$ 3P lag model. These figures have been plotted for the ramping parameter $t_0 = 75, 76, 77$ respectively. From these figures, we see that oscillatory natures in the propagation of ϕ and θ are observed. Further, the peak of oscillation will increase with the increase of t_0 and finally it diminishes to zero on the upper boundary of the beam.

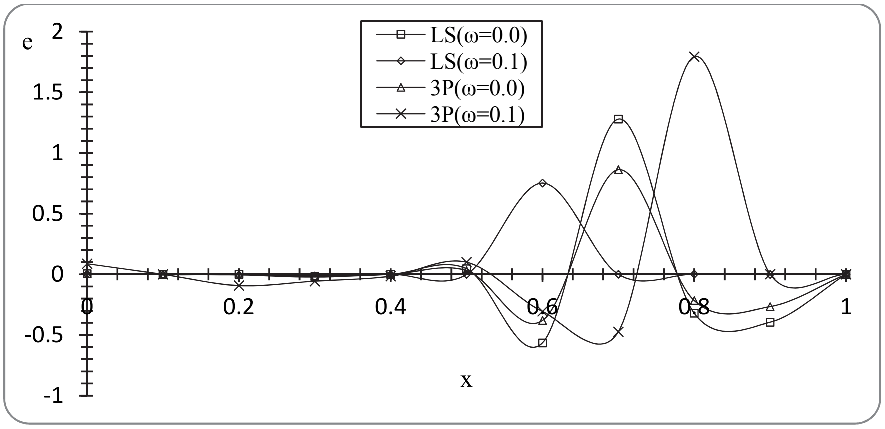

Fig. 5. Variation of e versus x for $\omega = 0.0, 0.1$ and $t_0 = 75$

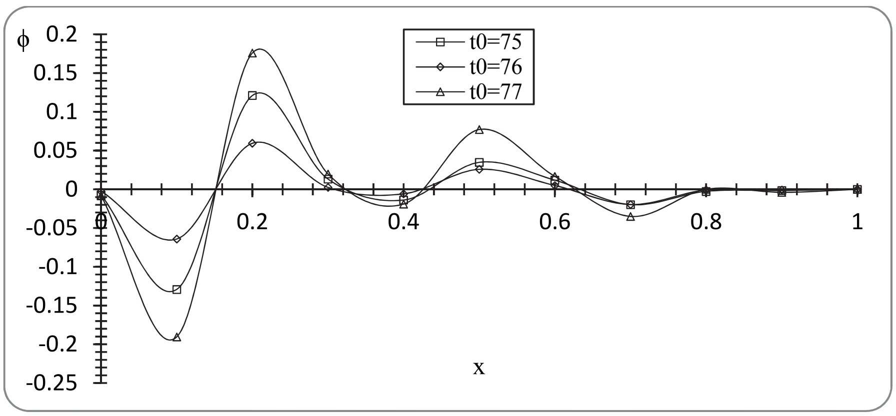

Fig. 6. Variation of ϕ versus x for $\omega = 0.1$ and $t_0 = 75, 76, 77$

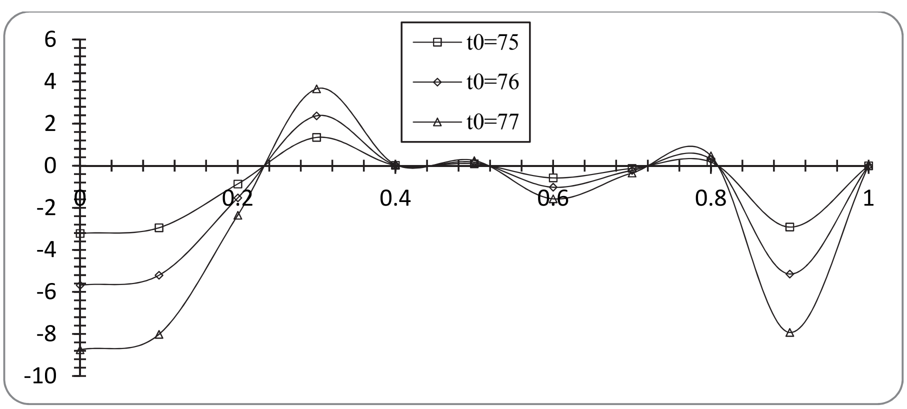

Fig. 7. Variation of θ versus x for $\omega = 0.1$ and $t_0 = 75, 76, 77$

7. Conclusions

The problem of investigating the lateral vibration, conductive temperature, thermodynamic temperature, thermoelastic stress and strain in a gold-nano beam in the light of two-temperature generalized thermoelasticity employing the three-phase-lag model and Lord-Shulman model of heat conduction. The method of Laplace transform is used to write the basic equations in the form of vector-matrix differential equation, which is then solved by Eigen-value approach. The numerical inversion of Laplace transform is done by using Fourier series expansion technique (Honig and Hirdes, 1984). All the figures plotted are self-explanatory in exhibiting the different peculiarities which occur in the propagation of waves, yet the following remarks may be added.

1. The significant differences in the physical quantities are observed for all the one-temperature model and two-temperature models. Two-temperature theory is more realistic than the one-temperature theory in the case of generalized thermoelasticity.

2. From the figures, it is seen that the ramping parameter have a significant effect on the studied fields. Whenever the ramp-type pulse is given to such a beam problem, the effect of the ramp-type parameter should be taken into consideration.

3. Here, all the results for one temperature theory agree with the existing literature (Sur and Kanoria, 2014b).

4. As a final remark, the results presented in this paper should prove useful for researchers in material science, designers of new materials, low temperature physicists as well as for those working on the development of a theory of hyperbolic thermoelasticity.

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