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# Restrictions on the stress components in the edge points of the homogeneous elastic body

V. Pestrenin[a*], I. Pestrenina[a] and L. Landik[a]

[a]Perm State National Research University, Perm, Russia

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ABSTRACT

The concept of a deformable body point is taken in the form of a continuum point and an elementary volume connected with it. The continuum point determines the spatial position of the deformable body point, and the elementary volume is the carrier of its material properties and stress-strain state characteristics. Based on this representation, restrictions on the state parameters at the edge points (singular points) of an isotropic body are constructed. These restrictions become preset conditions in singular points. The study covers possible interactions of the edge-forming surfaces with the external environment. It is shown that the restrictions formed at the edge points are usually more numerous than the restrictions at the regular point on the surface of a deformable body. This circumstance leads to a non-classical formulation of the mechanics problem for bodies with singular points. The combinations of geometric and material parameters of a structural component are discovered that determine the singular stresses behavior in elementary volumes with edge points. The load restrictions ensuring the compatibility of parameters defined at singular points are formulated. The attained results will apply to studying stress concentration in the vicinity of 3D-edges of structural components.



## 1. Introduction

The singular points of elastic bodies are the tips of cracks, wedges, cones, polyhedrons, edge points of the joints surfaces, intersection lines of the forming surfaces (edges), etc. Singular points are potential stress concentrators, destruction is born near them. Therefore, the study of the behavior features of state parameters (stresses and strains) of a body in the vicinity of such points is relevant and a large number of publications are devoted to it. At present, two approaches are used to study stress fields in the singular points vicinity. The first one (hereinafter classical or asymptotic) is characterized by exclusion a singular point from consideration by placing the pole of a curvilinear coordinate system into it (there is no one-to-one correspondence between a point of the body and its coordinates in the pole; therefore, this point is not included in the solution construction area, only asymptotic values of the desired parameters make sense in it).The exception of a singular point from the region of constructing a solution in the classical approach leads to an inadequate determination of stresses in its small neighborhood, since it does not consider the conditions (for example, boundary conditions) specified directly at the point itself. Publications reviews on the classical approach application are given in (Sinclair 2004, Paggi and Carpinteri 2008). The solution in the classical case is constructed by various methods: operational calculus (Williams, 1952, Cook & Erdogan, 1972, Sinclair, 2004), functions of a complex variable

* Corresponding author.
E-mail addresses: pestreninvm@mail.ru (V. Pestrenin)


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(Parton & Perlin, 1981), Erie functions and integral equations (Cook & Erdogan, 1972; Andreev, 2014), separation of variables and expansion in series into various functions (Shannon et al., 2014, 2015; Galadzhiev et al., 2011; He & Kotousov 2016), etc. The authors who are using numerical methods: finite element method (Koguchi & Muramoto, 2000; Barut et al., 2001; Xu & Sengupta, 2004; Lee et al., 2006; Xu et al., 2016; Dimitrov et al., 2001), finite element method in combination with by searching for eigenvalues by the Arnold method (Apel et al., 2002), the method of boundary elements and the method of boundary states (Mittelstedt & Becker, 2006; Koguchi & Da Costa, 2010 ), implementing the asymptotic idea by unlimited refinement of the FE-grid at the region near the special points or by constructing special finite elements. Many authors of asymptotic solutions in the study of the stress state near singular points (Williams, 1952; Koguchi & Muramoto, 2000; Wu, 2006, Koguchi & Da Costa, 2010; He & Kotousov, 2016) are looking for singularity indices – parameters for solving the characteristic equations of the corresponding homogeneous problems. It is believed that when certain criteria are formulated for such parameters, the solution for stresses tends to infinity as the distance to the singular point tends to zero. Such decisions, in our opinion, are not complete, since the criteria applied by those authors are not examined for the sufficiency of the fulfillment of the declared fact. There are examples when the criterion is satisfied, and unlimited growth of stresses near the singular point does not occur. The second (non-classical) approach overcomes the disadvantages of asymptotic research methods mentioned above. In an alternative approach, according to the idea of a continuous medium point developed by 18th century scientists (such as Bernoulli, D'Alembert and Euler,) and recognized by modern researchers, a special point (like any other point deformable body) is considered in the form of a small particle, which is a point of the continuum and the associated with it elementary volume at the same time. The point of the continuum indicates the location of the particle, and the elementary volume is responsible for its material characteristics and the stress-strain state. The elementary volume has a characteristic (linear) size equal to the characteristic size of the representative volume of the modeled body. The state parameters of an elementary volume are homogeneous, since they are averaged over a representative body volume by the values of lower structural level parameters. As a result, for the constraints specified at the singular point are taken the constraints given for the elementary volume containing this point. For the first time, such an approach to the study of state parameters at a particular point and its surroundings was used by authors earlier, where it was shown that the singularity (uniqueness) of a particular point manifests itself is an excessive amount (compared to the usual boundary point) of the constraints specified in it. This circumstance makes the problem of mechanics of a deformable solid with a non-classical singular point. Non-classical (in the indicated sense) problems were considered in the subsequent works of these authors such as (homogeneous flat wedges (Pestrenin et al., 2016), composite flat wedges (Pestrenin et al., 2015, 2017), composite spatial edges, internal singular points in flat and spatial structural elements (Pestrenin et al., 2018), three-dimensional composite edges (Pestrenin & Pestrenina, 2017), circular homogeneous and composite cones, regular polyhedral (Pestrenin et al., 2014). In the present article, a non-classical approach is used to study the state parameters at the edge of a homogeneous elastic body. Various external influences on this edge are being studied. The preset restrictions for stress and strain components at the edge points depend on a combination of geometrical, material, and load parameters. The corresponding statements of the solid mechanics problems are determined. A solution to the problem of stretching a structural component consisting of two truncated cones is presented. The decisions obtained by the classical and non-classical approach are compared.

## 2. Investigating the restrictions on the state parameters at the edge point

### *2.1. Statement of the problem*

By a 3D-edge $\Gamma$ we mean a line formed by the intersection of two different boundary surfaces 1, 2 of the construction elements. At any point $A$ of the edge $\Gamma$ we construct its normal section (Fig.1) and introduce an orthonormal basis $\overline{r}_1, \overline{r}_2, \overline{r}_3$.

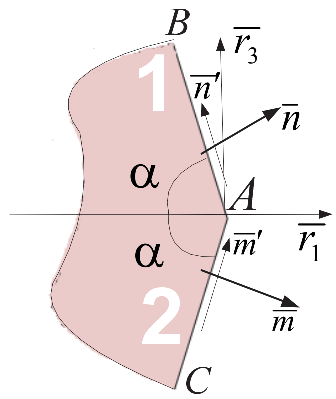


**Fig. 1.** Normal section of rib Γ

Vector $\overline{r}_1$ is directed to the outside of the body by the bisector of the angle $2\alpha$ ($0 < \alpha \le \pi$). Angle is formed in the normal plane by tangents at the point *A* to the lines, obtained as the intersection of this plane with the surfaces forming the edge. Vector $\overline{r}_2$ - along the tangent to the edge Γ, and the unit vector $\overline{r}_3$ in such a way that the vectors $\overline{r}_1, \overline{r}_2, \overline{r}_3$ form a right triple. We associate the Cartesian coordinate system A$x_1, x_2, x_3$ with the constructed vectors. It is assumed that the vectors $\overline{n}, \overline{m}$, restored to the surfaces 1 and 2 at the point *A* belong to the normal plane of the edge Γ. We denote the units $\overline{n}', \overline{m}'$ vectors perpendicular to the vectors $\overline{n}, \overline{m}$ respectively and to the vector $\overline{r}_2$. The following notations are adopted: $\sigma_{ij}$ - components of the stress tensor; $\varepsilon_{ij}$ - components of the strain tensor; $\sigma_n$ - normal stress on the surface 1; $\tau_{n'}$, $\tau_{n2}$ – are the components of the tangential stress vector on the surface 1 along the directions of $\overline{n}'$ and $\overline{r}_2$; $\sigma_m$ - normal stress on the surface 2; $\tau_{m'}$, $\tau_{m2}$ are the components of the tangential stress vector on the surface 2 along the directions of $\overline{m}'$ and $\overline{r}_2$; $\overline{p}_n = p_n\overline{n} + \tau_n\overline{n}' + \xi_n\overline{r}_2$ - given stress vector on surface 1; $\overline{p}_m = p_m\overline{m} + \tau_m\overline{m}' + \xi_m\overline{r}_2$ - a given stress vector on the surface 2; *E* - Young's modulus, *G* - shear modulus, ν - Poisson's ratio, ω - temperature deformation coefficient, $\Delta T$ - temperature increment.

In accordance with the accepted concept, the state parameters at the points of the rib Γ (special points) are identified with the state parameters of the elementary volumes containing such points. In this regard, the task is to formulate restrictions on the state parameters of elementary volumes adjacent to the edge Γ, for various interactions of the surfaces forming it with the external environment and to study the influence of the material and geometric parameters of the structural elements and loading options on such restrictions.

## 3. Forming surfaces of wedge are free from loads

It is assumed that surface loads on the forming surfaces of the edge Γ (near point *A*) are absent (Fig. 1). Then in the elementary volume containing point A on the sites oriented by the vectors $\overline{n}, \overline{m}$, the conditions are met

$$\sigma_n = 0, \tau_{n'} = 0, \tau_{n2} = 0, \sigma_m = 0, \tau_{m'} = 0, \tau_{m2} = 0 \tag{3.1}$$

Through the stresses, Eqs. (3.1) are written by two autonomous systems of equations

$$\sigma_{11}\sin^2\alpha + 2\sigma_{13}\sin\alpha\cos\alpha + \sigma_{33}\cos^2\alpha = 0$$
$$-\sigma_{11}\sin\alpha\cos\alpha + \sigma_{13}(\sin^2\alpha - \cos^2\alpha) + \sigma_{33}\sin\alpha\cos\alpha = 0$$

$$\sigma_{11}\sin^2\alpha - 2\sigma_{13}\sin\alpha\cos\alpha + \sigma_{33}\cos^2\alpha = 0 \qquad (3.2)$$
$$\sigma_{11}\sin\alpha\cos\alpha + \sigma_{13}(\sin^2\alpha - \cos^2\alpha) - \sigma_{33}\sin\alpha\cos\alpha = 0$$

$$\sigma_{12}\sin\alpha + \sigma_{32}\cos\alpha = 0 \qquad (3.3)$$
$$\sigma_{12}\sin\alpha - \sigma_{32}\cos\alpha = 0$$

Four different determinants of the third order $\Delta_i$ $(i = 1,2,3,4)$ are constructed from the matrix of the system of four Eqs. (3.2) with the three unknowns $\sigma_{11}, \sigma_{13}, \sigma_{33}$

$$\Delta_1 = \Delta_2 = \sin^2 2\alpha, \qquad \Delta_3 = -\Delta_4 = \cos 2\alpha \sin 2\alpha, \qquad (3.4)$$

which simultaneously vanish at the points:

$$\alpha = \pi/2, \qquad \alpha = \pi \qquad (3.5)$$

The determinant of the system of Eqs. (3.3) $\Delta = -\sin 2\alpha$ also vanishes at the points (3.5). The following cases of solving systems of Eqs. (3.2), (3.3) are possible:

1) $\alpha \neq \pi/2, \quad \alpha \neq \pi$. The determinants of homogeneous systems (3.2), (3.3) are not equal to zero, so their solution $\sigma_{11} = 0, \sigma_{12} = 0, \sigma_{13} = 0, \quad \sigma_{23} = 0, \ \sigma_{33} = 0$. These stress values are given independent restrictions at the edge points. The number of restrictions is redundant, equal to five. In the classical problem, the number of restrictions in the ordinary (not special) point of the surface is three.

2) $\alpha = \pi/2$. From Eqs. (3.2), (3.3) we obtain $\sigma_{11} = 0, \sigma_{13} = 0, \sigma_{12} = 0$. These three independent restrictions imposed on the stresses at the material point *A*. The number of restrictions corresponds to the classical case. *A* special point loses its status, becomes an ordinary point on the body surface.

3) $\alpha = \pi$. From the Eqs. (3.2), (3.3) it follows that $\sigma_{33} = 0, \sigma_{13} = 0, \sigma_{32} = 0$.
The number of restrictions corresponds to the classical problem.

**4. Forming surfaces of wedge are loaded by surface forces**

Suppose that the forming surfaces of the edge $\Gamma$ in the neighborhood of the point A are loaded by the stress vectors $\overline{p}_n$ and $\overline{p}_m$. Then at the material point *A* we can write the equalities

$$\sigma_n = p_n, \qquad \tau_{n'} = \tau_n, \quad \tau_{n2} = \xi_n, \quad \sigma_m = p_m, \qquad \tau_{m'} = \tau_m, \qquad \tau_{m2} = \xi_m.$$

Recorded through the stresses, these equalities form two autonomous linear inhomogeneous systems of equations for the stresses $\sigma_{11}, \sigma_{13}, \sigma_{33}$ and $,\sigma_{12}, \sigma_{32}$:

$$\sigma_{11}\sin^2\alpha + 2\sigma_{13}\sin\alpha\cos\alpha + \sigma_{33}\cos^2\alpha = p_n$$
$$-\sigma_{11}\sin\alpha\cos\alpha + \sigma_{13}(\sin^2\alpha - \cos^2\alpha) + \sigma_{33}\sin\alpha\cos\alpha = \tau_n$$
$$\sigma_{11}\sin^2\alpha - 2\sigma_{13}\sin\alpha\cos\alpha + \sigma_{33}\cos^2\alpha = p_m \qquad (4.1)$$
$$\sigma_{11}\sin\alpha\cos\alpha + \sigma_{13}(\sin^2\alpha - \cos^2\alpha) - \sigma_{33}\sin\alpha\cos\alpha = \tau_m$$

$$\sigma_{12}\sin\alpha + \sigma_{32}\cos\alpha = \xi_n \qquad (4.2)$$
$$\sigma_{12}\sin\alpha - \sigma_{32}\cos\alpha = \xi_m$$

The matrices of the systems of Eqs. (4.1), (4.2) coincide respectively with the matrices of the systems of Eqs. (3.2), (3.3), therefore the following solutions cases of Eqs. (4.1), (4.2) are possible:

1) $\alpha \neq \pi/2$, $\alpha \neq \pi$. The rank of the matrix of the system of Eq. (4.1) is three. In order for a solution to exist in this case, the rank of the extended matrix of the system must also be equal to three. This condition leads to equality

$$(p_n - p_m)\cos 2\alpha + (\tau_n + \tau_m)\sin 2\alpha = 0 \quad (4.3)$$

Eq. (4.3) is a restriction on the surface loads at the points of the edge $\Gamma$. It requires the matching of the applied load with the condition of stress tensor symmetry: the projection of the stress vector on the platform oriented by the vector $\vec{n}$ on the direction $\vec{m}$ must be equal to the projection of the stress vector on the platform oriented by the vector $\vec{m}$ on the direction $\vec{n}$ $(\vec{P}_n \cdot \vec{m} = \vec{P}_m \cdot \vec{n})$. When the restriction (4.3) is satisfied, equations (4.1) have a unique solution

$$\begin{aligned} \sigma_{11} &= \frac{1}{2}(p_n + p_m) + \frac{1}{2}(\tau_m - \tau_n)\operatorname{ctg}\alpha \\ \sigma_{13} &= \frac{1}{2}(\tau_m + \tau_n) + \frac{1}{2}(p_n - p_m)\operatorname{ctg}\alpha \\ \sigma_{33} &= \frac{1}{2}(p_n + p_m) + \frac{1}{2}(\tau_n - \tau_m)\operatorname{tg}\alpha \end{aligned} \quad (4.4)$$

If the restriction (4.3) does not hold, there is no solution within the framework of a symmetric theory. The material point *A* is the point of the singular behavior of the stress components $\sigma_{11}$, $\sigma_{13}$, $\sigma_{33}$. Therefore, condition (4.3) determines the combination of geometric parameters and load parameters, which ensures that the solution does not have a singular behavior at the points of the edge. The rank of the system of Eq. (4.2) is equal to two, it has a unique solution

$$\sigma_{12} = \frac{\xi_n + \xi_m}{2\sin\alpha} \qquad \sigma_{32} = \frac{\xi_n - \xi_m}{2\sin\alpha} \quad (4.5)$$

It is clear from (4.4), (4.5) that at the points of the edge $\Gamma$ under the conditions (4.3) the preset number of restrictions exceeds the number of restrictions at the surface point in the classical case.

2) $\alpha = \pi/2$. The compatibility condition for equations (4.1), (4.2) is the constraint on the load

$$p_n = p_m, \quad \tau_n = \tau_m, \quad \xi_n = \xi_m. \quad (4.6)$$

When the equalities (4.6) are satisfied, $\sigma_{11} = p_n$, $\sigma_{13} = \tau_n$, $\sigma_{12} = \xi_n$ follow from (4.1), (4.2). At the material point *A* three independent conditions are given, which corresponds to the classical case. *A* special point becomes an ordinary point of the surface.

3) $\alpha = \pi$. Equations (4.1), (4.2) are compatible under constraints on the load

$$p_n = p_m, \quad \tau_n = \tau_m, \quad \xi_n = -\xi_m. \quad (4.7)$$

In this case $\sigma_{33} = p_n$, $\sigma_{13} = -\tau_n$, $\sigma_{23} = -\xi_n$. Such a number of constraints at the material point *A* corresponds to the classical problem.

The constraints on the load (4.3), (4.4), (4.7) are of the same nature - they are the conditions for the solution existence to the problem of preset constraints at a material point *A* in the framework of a symmetric stress theory. Failure to comply with these restrictions leads to a singularity nature of the state parameters at this point.

**4. An edge with a rigidly clamped surface and a free surface**

It is assumed that the forming surface 1 is free from the load, and the forming surface 2 is rigidly clamped. From this it follows that at the material point *A* of the edge, on the area oriented by the vector $\bar{n}$, the normal and tangential stresses vanish

$$\sigma_n = 0, \qquad \tau_{n'} = 0, \quad \tau_{n2} = 0. \tag{5.1}$$

On an area oriented by an vector $\bar{m}$, the relative elongations of any fibers and the shifts between any directions are equal to zero, which is equivalent to the relations

$$\varepsilon_{ij} m'_i m'_j = 0, \qquad \varepsilon_{ij} n_{2i} n_{2j} = 0, \qquad \varepsilon_{ij} m'_i n_{2j} = 0. \tag{5.2}$$

The Eqs. (5.1), (5.2), expressed in terms of stresses, for a linear thermoelastic body are written down by two autonomous systems of equations

$$\begin{aligned}
&\sigma_{11}\sin^2\alpha + 2\sigma_{13}\sin\alpha\cos\alpha + \sigma_{33}\cos^2\alpha = 0 \\
&-\sigma_{11}\sin\alpha\cos\alpha + \sigma_{13}(\sin^2\alpha - \cos^2\alpha) + \sigma_{33}\sin\alpha\cos\alpha = 0 \\
&\sigma_{11}(\cos^2\alpha - \nu\sin^2\alpha) + (1+\nu)\sigma_{12}\sin 2\alpha + \sigma_{33}(\sin^2\alpha - \nu\cos^2\alpha) - \nu\sigma_{22} = -\omega E\Delta T \\
&-\nu\sigma_{11} - \nu\sigma_{33} + \sigma_{22} = -\omega E\Delta T
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
&\sigma_{12}\sin\alpha + \sigma_{32}\cos\alpha = 0 \\
&\sigma_{12}\cos\alpha + \sigma_{32}\sin\alpha = 0
\end{aligned} \tag{5.4}$$

The determinant $\Delta = \cos^2 2\alpha - \nu$ of the system of Eqs. (5.3) for values $\nu$ in the interval (0; 0.5) vanishes at four points $\alpha_i, (i = 1,2,3,4)$:

$$\begin{aligned}
&\sin\alpha_1 = \sqrt{\frac{1-\sqrt{\nu}}{2}} \qquad \cos\alpha_1 = \sqrt{\frac{1+\sqrt{\nu}}{2}}, (0 < \alpha_1 < \pi/4) \\
&\sin\alpha_2 = \sqrt{\frac{1-\sqrt{\nu}}{2}} \qquad \cos\alpha_2 = -\sqrt{\frac{1+\sqrt{\nu}}{2}}, (3\pi/4 < \alpha_2 < \pi) \\
&\sin\alpha_3 = \sqrt{\frac{1+\sqrt{\nu}}{2}} \qquad \cos\alpha_2 = \sqrt{\frac{1-\sqrt{\nu}}{2}}, (\pi/4 < \alpha_3 < \pi/2) \\
&\sin\alpha_4 = \sqrt{\frac{1+\sqrt{\nu}}{2}} \qquad \cos\alpha_4 = -\sqrt{\frac{1-\sqrt{\nu}}{2}}, (\pi/2 < \alpha_4 < 3\pi/4)
\end{aligned} \tag{5.5}$$

The determinant of the matrix of the system of Eqs. (5.4) vanishes at points

$$\alpha = \pi/4 \qquad \alpha = 3\pi/4 \tag{5.6}$$

Possible cases of solutions of Eqs (5.3), (5.4) are

1) The angle $\alpha$ is not defined by the Eqs. (5.5) or (5.6), the rank of the matrix of the system of Eqs. (5.3) is equal to four. It has a unique solution.

$$\sigma_{11} = -\frac{\omega E \Delta T \cos^2\alpha}{\cos^2 2\alpha - \nu} \qquad \sigma_{13} = -\frac{\omega E \Delta T \sin\alpha\cos\alpha}{\cos^2 2\alpha - \nu} \qquad (5.7)$$

$$\sigma_{33} = -\frac{\omega E \Delta T \sin^2\alpha}{\cos^2 2\alpha - \nu} \qquad \sigma_{22} = \frac{\omega E \Delta T \cos^2 2\alpha}{\cos^2 2\alpha - \nu}$$

Eqs. (5.4) have a trivial solution

$$\sigma_{12} = 0 \qquad \sigma_{32} = 0 \qquad (5.8)$$

The number of constraints (5.7), (5.8) imposed at the material point *A* exceeds the number of constraints given at the surface point in the classical problem.

2) Let $\alpha$ take one of the values (5.5). These values (if $\nu \neq 0$) do not coincide with the points (5.6). In the absence of a temperature load $(\Delta T = 0)$, the system of equations (5.3) is consistent. Between the stresses for any of $\alpha_i, (i = 1,2,3,4)$ the relations on the stresses are valid $\sigma_{11} = \sigma_{33} ctg^2\alpha$, $\sigma_{13} = -\sigma_{33} ctg\,\alpha$, $\sigma_{22} = \nu\sigma_{33} / \sin^2\alpha$. And $\sigma_{12}$ , $\sigma_{32}$ are determined by the equalities (5.8). The number of constraints set at the edge points exceeds the constraints number at the boundary surface points in the classical problem.

In the presence of a temperature load ($\Delta T \neq 0$), the rank of the extended matrix of the Eqs. system (5.3) turns out to be equal to four, which is greater than the rank of the matrix of the system. Therefore the system of equations (5.3) is inconsistent. This means that in the case when the angle tends to any of the values $\alpha_i, (i = 1,2,3,4)$, the stresses $\sigma_{11}, \sigma_{13}, \sigma_{33}, \sigma_{22}$ at the material point *A* increase indefinitely. Consequently, the combination of the geometric and material parameters of the deformed body, determined by the Eqs. (5.5), is critical for the temperature load.

3) $\alpha = \pi/4$. The following restrictions are imposed on the stresses in the material point A: the solution (5.7) of the system of Eqs. (5.3) and the dependence between the stresses $\sigma_{12}, \sigma_{32}$ : $\sigma_{12} = -\sigma_{32}$, arising from Eqs. (5.4). The number of restrictions exceeds the number of constraints at the boundary points in the classical case.

4) $\alpha = 3\pi/4$. At the material point *A*, Eqs. (5.3) have a solution (5.7). Eqs. (5.4) implies the dependence $\sigma_{12} = \sigma_{32}$. The number of constraints at the special point exceeds the number of constraints at the boundary points in the classical problem.

## 5. An edge with a rigidly clamped surface and a loaded surface

It is assumed that the forming surface 1 near the edge vertex is loaded with surface force $\overline{p}_n$, and the forming surface 2 is rigidly clamped. These restrictions, using the Eqs. (5.2) established in the previous section for a linearly elastic body, in stresses are written down by two systems of equations

$$\begin{gathered}
\sigma_{11}\sin^2\alpha + 2\sigma_{13}\sin\alpha\cos\alpha + \sigma_{33}\cos^2\alpha = p_n \\
-\sigma_{11}\sin\alpha\cos\alpha + \sigma_{13}(\sin^2\alpha - \cos^2\alpha) + \sigma_{33}\sin\alpha\cos\alpha = \tau_n \\
\sigma_{11}(\cos^2\alpha - \nu\sin^2\alpha) + (1+\nu)\sigma_{12}\sin 2\alpha + \sigma_{33}(\sin^2\alpha - \nu\cos^2\alpha) - \nu\sigma_{22} = 0 \\
-\nu\sigma_{11} - \nu\sigma_{33} + \sigma_{22} = 0
\end{gathered} \qquad (6.1)$$

$$\sigma_{12}\sin\alpha+\sigma_{32}\cos\alpha=\xi_n$$
$$\sigma_{12}\cos\alpha+\sigma_{32}\sin\alpha=0 \tag{6.2}$$

The determinant of the matrix of the system of Eqs. (6.1) $\Delta=\cos^2 2\alpha-\nu$ vanishes in the interval $0<\alpha\le\pi$ for the values $\alpha_i,(i=1,2,3,4)$ determined by Eqs. (5.5). The determinant of the system of equations (6.2) vanishes at the points (5.6). There are possible cases of solutions of Eq. (6.1) and Eq. (6.2).

1) The angle $\alpha$ does not coincide with any of the angles $\alpha_i,(i=1,2,3,4)$ determined by formulas (5.5), (5.6). The determinants of the systems of Eq. (6.1), Eq. (6.2) do not vanish. Then the Eq. (6.1) and Eq. (6.2) have a unique solution

$$\begin{aligned}
\sigma_{11}&=\frac{p_n[(\nu-\sin^2\alpha)\cos 2\alpha-0.5\sin^2 2\alpha]+\tau_n[\cos 2\alpha-\nu]\sin 2\alpha}{\cos^2 2\alpha-\nu}\\
\sigma_{13}&=\frac{p_n(1-2\nu)\sin\alpha\cos\alpha+\tau_n(\nu-1)\cos 2\alpha}{\cos^2 2\alpha-\nu}\\
\sigma_{33}&=\frac{p_n[(\cos^2\alpha-\nu)\cos 2\alpha-0.5\sin^2 2\alpha]+\tau_n[\cos 2\alpha-\nu]\sin 2\alpha}{\cos^2 2\alpha-\nu}\\
\sigma_{22}&=\frac{p_n\nu\cos 4\alpha+\tau_n\nu\sin 4\alpha}{\cos^2 2\alpha-\nu}\quad \sigma_{12}=-\frac{\xi_n\sin\alpha}{\cos 2\alpha},\quad \sigma_{23}=\frac{\xi_n\cos\alpha}{\cos 2\alpha}.
\end{aligned} \tag{6.3}$$

Six constraints are imposed on the stresses at the special point (6.3). Such a number of constraints exceed their numbers at usual surface point in the classical case.

2) If the angle $\alpha$ coincides with any of the angles (5.5), the rank of the system of equations (6.1) becomes equal to three. In order to calculate the rank of the extended matrix, four determinants are constructed, obtained by successively replacing the columns of the matrix of the system (6.1) by the free column

$$\Delta_i=p_n f_i(\nu)+\tau_n\varphi_i(\nu),\quad (i=1,2,3,4) \tag{6.4}$$

Herein the following functions occur:

$$f_1(\nu)=(\nu-\sin^2\alpha)\cos 2\alpha-0.5\sin^2 2\alpha \qquad \varphi_1(\nu)=(\nu+\cos 2\alpha)\sin 2\alpha$$
$$f_2(\nu)=(1-2\nu)\sin\alpha\cos\alpha \qquad \varphi_2(\nu)=(\nu-1)\cos 2\alpha$$
$$f_3(\nu)=(\cos^2\alpha-\nu)\cos 2\alpha-0.5\sin^2 2\alpha \qquad \varphi_3(\nu)=(\cos 2\alpha-\nu)\sin 2\alpha$$
$$f_4(\nu)=\nu\cos 4\alpha \qquad \varphi_4(\nu)=\nu\sin 4\alpha$$

The formulation (6.4) for the determinants $\Delta_i(i=1,2,3,4)$ is valid for each angle $\alpha_i,(i=1,2,3,4)$ defined by Eq. (5.5). The rank of the extended matrix of the system of Eqs. (6.1) will be equal to three if all four determinants (6.4) for the angle $\alpha_n,(n=1,2,3,4)$ vanish. This requirement leads to the following restrictions on the load

$$p_n(2\nu-1)+2\tau_n\sqrt{\nu(1-\nu)}=0,\quad for\quad \alpha_1,\alpha_4$$
$$p_n(1-2\nu)+2\tau_n\sqrt{\nu(1-\nu)}=0,\quad for\quad \alpha_2,\alpha_3 \tag{6.5}$$

When the constraints (6.5) are satisfied, the rank of the extended matrix of equations (6.1) is three. Between the stress components, the following dependencies are fulfilled

$$\sigma_{11} = -p_n(c\,\mathrm{tg}^2\,\alpha - 1) - 2\tau_n ctg\alpha + \sigma_{33} c\,\mathrm{tg}^2\,\alpha \qquad \sigma_{13} = (p_n - \sigma_{33})\,\mathrm{ctg}\,\alpha + \tau_n$$

$$\sigma_{22} = -\nu(p_n \cos 2\alpha + \tau_n \sin 2\alpha - \sigma_{33}) / \sin^2\alpha$$

In these equalities, the angle $\alpha$ takes the values (5.5) $\alpha_i, (i = 1,2,3,4)$. The stresses $\sigma_{12}$ and $\sigma_{23}$ in this case are given by Eq. (6.3). The number of constraints on the parameters $\sigma_{ij}$ at the special point exceeds the restrictions number at the boundary points of the classical problem. If the restrictions (6.5) are not satisfied, the rank of the extended matrix of the system (6.1) is greater than the rank of the system matrix, its solution does not exist. Therefore, Eqs. (6.5) are the conditions for the absence of a stress singularity at a special point under surface loading $\overline{p}_n$.

3) $\alpha = \pi/4$. From equations (6.1) we find

$$\sigma_{11} = \frac{1}{2\nu} p_n - \tau_n \qquad \sigma_{13} = -\frac{1-2\nu}{2\nu} p_n$$

$$\sigma_{33} = \frac{1}{2\nu} p_n + \tau_n \qquad \sigma_{22} = p_n$$

The compatibility condition for Eqs. (6.2) is the restriction on the load

$$\xi_n = 0. \tag{6.6}$$

The stresses $\sigma_{12}$ and $\sigma_{23}$ are dependent $\sigma_{12} = -\sigma_{32}$. The state parameters at the special point are superimposed with an excessive number of restrictions in comparison with the usual point of the body.

4) $\alpha = 3\pi/4$. Eqs. (6.1) have the solution

$$\sigma_{11} = \frac{1}{2\nu} p_n + \tau_n \qquad \sigma_{13} = \frac{1-2\nu}{2\nu} p_n$$

$$\sigma_{33} = \frac{1}{2\nu} p_n - \tau_n \qquad \sigma_{22} = p_n$$

The compatibility condition for Eqs. (6.2) is the restriction on the load (6.6). The stresses $\sigma_{12}$ and $\sigma_{23}$ are connected by a dependence $\sigma_{12} = \sigma_{32}$. The number of independent constraints set at a special point is redundant.

## 4. AN EDGE WITH A FREE SURFACE AND A FRICTIONLESS SLIDING SURFACE

Suppose that the forming surface 1 near the point $A$ is free from the load, and the forming surface 2 slides without friction along a rigid surface. At the material point $A$, the displacements in the direction of the vector $\overline{m}$ vanish and the surface loads are constrained $\sigma_n = 0$; $\tau_{n'} = 0$; $\tau_{n2} = 0$; $\tau_{m'} = 0$; $\tau_{m2} = 0$. These equations, in terms stress tensor components, form two autonomous systems of equations

$$\begin{aligned} &\sigma_{11} \sin^2\alpha + 2\sigma_{13} \sin\alpha\cos\alpha + \sigma_{33}\cos^2\alpha = 0 \\ &-\sigma_{11}\sin\alpha\cos\alpha + \sigma_{13}(\sin^2\alpha - \cos^2\alpha) + \sigma_{33}\sin\alpha\cos\alpha = 0 \\ &\sigma_{11}\sin\alpha\cos\alpha - \sigma_{13}\cos 2\alpha - \sigma_{33}\sin\alpha\cos\alpha = 0 \end{aligned} \tag{7.1}$$

$$\begin{aligned} &\sigma_{12}\sin\alpha + \sigma_{32}\cos\alpha = \xi_n \\ &\sigma_{12}\sin\alpha - \sigma_{32}\cos\alpha = 0 \end{aligned} \tag{7.2}$$

The determinant $\Delta = \sin 2\alpha \cos 2\alpha$ of the system of equations (7.1) vanishes on the interval $0 < \alpha \leq \pi$ at the points

$$\alpha = \pi/4 \quad \alpha = \pi/2 \quad \alpha = 3\pi/4 \quad \alpha = \pi \tag{7.3}$$

The determinant $\Delta' = \sin 2\alpha$ of the system of equations (7.2) vanishes at the points

$$\alpha = \pi/2 \qquad \alpha = \pi \tag{7.4}$$

which are included in the list (7.3). The following cases of solutions of equations (7.1), (7.2) are possible:

1) $\alpha \neq \pi/4, \alpha \neq \pi/2, \alpha \neq 3\pi/4, \alpha \neq \pi.$ The determinants of the systems of equations (7.1), (7.2) do not vanish, the equation has only the trivial solution $\sigma_{11} = \sigma_{13} = \sigma_{33} = \sigma_{12} = \sigma_{32} = 0$. In total, five independent constraints on the components of the stress tensor at the material point *A* are given.

2) $\alpha = \pi/4$. The rank of the matrix of the system of equations (7.1) is equal to two, so the two stresses depend on the third one $\sigma_{11} = \sigma_{33}$; $\sigma_{13} = -\sigma_{33}$. The determinant of system (7.2) is nonzero, the solution of this system is trivial $\sigma_{12} = 0$; $\sigma_{23} = 0$. The number of restrictions at a special point exceeds the number of constraints at the boundary points in the classical problem.

3) $\alpha = \pi/2$. Two stresses $\sigma_{11} = 0$; $\sigma_{13} = 0$ are determined from equations (7.1). Equations (7.2) have a determinant equal to zero, from which it follows $\sigma_{12} = 0$. In total three independent constraints on the stress tensor components are given at the material point *A*. In addition, at the point *A*, the condition for displacement ($u_1 = 0$), which is not involved in the construction of these restrictions, has to specified.

4) $\alpha = 3\pi/4$. From equations (7.1) we find the dependences between the stresses $\sigma_{11} = \sigma_{33}$; $\sigma_{13} = \sigma_{33}$. Equations (7.2) have a solution $\sigma_{12} = 0$; $\sigma_{23} = 0$. The number of given conditions at a special point exceeds the number of constraints at the boundary points in the classical case.

5) $\alpha = \pi$. In this case, (as in unit 3), equations (7.1), (7.2) have a solution $\sigma_{33} = 0$; $\sigma_{13} = 0$; $\sigma_{23} = 0$. In addition to these conditions, it is necessary to take into account ( $u_3 = 0$ ).

**6. An Edge with a loaded surface and a frctionless sliding surface**

It is assumed that the forming 1 of the edge is loaded near the point *A* by surface load $\bar{p}_n$, and the forming two slides without friction along a rigid surface. The given conditions on the forces at the material point *A* may be written by inhomogeneous equations

$$\begin{aligned}
&\sigma_{11} \sin^2 \alpha + 2\sigma_{13} \sin\alpha \cos\alpha + \sigma_{33} \cos^2 \alpha = p_n \\
&-\sigma_{11} \sin\alpha \cos\alpha + \sigma_{13}(\sin^2 \alpha - \cos^2 \alpha) + \sigma_{33} \sin\alpha \cos\alpha = \tau_n \\
&\sigma_{11}(\cos^2 \alpha - \nu \sin^2 \alpha) + (1+\nu)\sigma_{12} \sin 2\alpha + \sigma_{33}(\sin^2 \alpha - \nu \cos^2 \alpha) - \nu\sigma_{22} = 0
\end{aligned} \tag{8.1}$$

$$\begin{aligned}
&\sigma_{12} \sin\alpha + \sigma_{32} \cos\alpha = \xi_n \\
&\sigma_{12} \cos\alpha + \sigma_{32} \sin\alpha = 0
\end{aligned} \tag{8.2}$$

The determinant $\Delta = \cos 2\alpha \sin 2\alpha$ of the matrix of system (8.1) vanishes at points (7.3): $\alpha_1 = \pi/4, \alpha_2 = \pi/2, \alpha_3 = 3\pi/4, \alpha_4 = \pi,$ and the determinant $\Delta' = \sin 2\alpha$ of the matrix of system (8.2) vanishes at points (7.4). Therefore, solutions are possible for the systems of Eqs. (8.1) and Eqs. (8.2) in cases:

1) $\alpha \neq \pi/4, \alpha \neq \pi/2, \alpha \neq 3\pi/4, \alpha \neq \pi$. Equations (8.1), (8.2) have a unique solution

$$\sigma_{11} = p_n + \frac{1}{2}\tau_n(tg2\alpha - ctg\alpha) \quad \sigma_{13} = -\frac{\tau_n}{2\cos 2\alpha} \quad \sigma_{33} = p_n + \frac{1}{2}\tau_n(tg\alpha + tg2\alpha)$$

$$\sigma_{12} = -\frac{\xi_n}{2\sin\alpha} \quad \sigma_{23} = \frac{\xi_n}{2\cos\alpha}$$

The number of constraints on the stress tensor components at the special point *A* exceeds the number of constraints given at any boundary points in the classical case.

2) $\alpha = \pi/4$. The rank of the matrix of the system of Eqs. (8.1) is equal to two, the compatibility condition for these equations is given by

$$\tau_n = 0, \tag{8.3}$$

expressing the law of the tangential stresses pairing in the symmetric theory of elasticity. When the restriction (8.3) is satisfied from Eqs. (8.1), we find $\sigma_{11} = \sigma_{33}, \quad \sigma_{13} = p_n - \sigma_{33}$.

Eqs. (8.2) have a solution $\sigma_{12} = \sigma_{32} = \frac{1}{\sqrt{2}}\xi_n$. The number of given conditions at the material point *A* is redundant in comparison with the classical case. If (8.3) does not hold, there is no solution of system (8.1). The special point turns out to be singular. When $\alpha$ tending to the value of $\pi/4$ the stress $\sigma_{11}, \sigma_{33}, \sigma_{13}$ at this point increase without limit.

3) $\alpha = \pi/2$. The restriction (8.3) is a condition for the compatibility of equations (8.1). When it is executed $\sigma_{11} = p_n, \quad \sigma_{13} = 0$. The rank of the system of equations (8.2) is equal to one. It is consistent under the condition

$$\xi_n = 0 \tag{8.4}$$

If the load satisfies the constraint (8.4), from equations (8.2), we obtain $\sigma_{12} = 0$. At a special point, three independent conditions on the stress components are given. There is also a restriction on the movement $(u_1 = 0)$. If the restrictions (8.3), (8.4) on the applied load are not satisfied, the solution of Eqs. (8.1) and Eqs (8.2) does not exist.

4) $\alpha = 3\pi/4$. The compatibility of Eqs. (8.1) is also ensured by (8.3). If it is satisfied, the equations between the stresses follow from equations (8.1) $\sigma_{11} = \sigma_{33}, \quad \sigma_{13} = \sigma_{33} - p_n$. Eqs. (8.2) have a solution $\sigma_{12} = \frac{1}{\sqrt{2}}\xi_n, \quad \sigma_{32} = -\frac{1}{\sqrt{2}}\xi_n$. The number of given conditions at material point *A* exceeds the constraints number at the surface points in the classical problem. If the restriction (8.3) to the applied load is not satisfied, the special point turns out to be singular.

5) $\alpha = \pi$. The compatibility condition for the system (8.1) and (8.2) is, respectively, the constraints (8.3) and (8.4). If they are satisfied, Eqs. (8.1) and Eqs. (8.2) have a solution $\sigma_{33} = p_n, \quad \sigma_{13} = 0, \quad \sigma_{32} = 0$.

At the special point, three conditions are imposed on the components of the stress tensor and the condition for displacement $(u_3 = 0)$.

## 7. An edge with two rigidly clamped surfaces

If the forming surfaces of the edge $\Gamma$ are under rigid pinching, the following conditions are superimposed on the state parameters at its any material point *A*:

a) equality to zero of the relative elongations of linear elements in the direction of the vectors $\overline{n}'$, $-\overline{m}'$, $\overline{r}_2$

b) zero shifts between linear elements whose directions are determined by pairs of vectors $(\overline{n}',\overline{r}_2)$, $(-\overline{m}',\overline{r}_2)$, $(\overline{n}',-\overline{m}')$.

These conditions in terms of the strain tensor components may be written by the equations

$$\begin{aligned}&\varepsilon_{11}\cos^2\alpha-2\varepsilon_{13}\sin\alpha\cos\alpha+\varepsilon_{33}\sin^2\alpha=0\\&\varepsilon_{11}\cos^2\alpha+2\varepsilon_{13}\sin\alpha\cos\alpha+\varepsilon_{33}\sin^2\alpha=0\\&\varepsilon_{11}\cos^2\alpha-\varepsilon_{33}\sin^2\alpha=0\end{aligned}\tag{9.1}$$

$$\begin{aligned}&-\varepsilon_{12}\cos\alpha+\varepsilon_{32}\sin\alpha=0\\&\varepsilon_{12}\cos\alpha+\varepsilon_{32}\sin\alpha=0\end{aligned}\tag{9.2}$$

$$\varepsilon_{22}=0\tag{9.3}$$

In constructing these equations, a formula is used that determines the shift $\varphi$ in an any point of a continuous medium between linear material elements with directions $\vec{k}$, $\vec{l}$ and the angle $\beta$ between them

$$\varphi\sin\beta=[2\varepsilon_{rp}-(\eta_k+\eta_l)\delta_{rp}]k_r\,l_p\tag{9.4}$$

In formula (9.4) there are denoted: $\eta_k$, $\eta_l$ –the relative elongations at the point of continuous medium in the direction of the vectors $\vec{k}$, $\vec{l}$, respectively, $\delta_{rp}$ –the coordinates of the metric tensor. The determinants of the systems of equations (9.1), (9.2) are respectively equal $\Delta_1=-\sin^3 2\alpha$, $\Delta_2=-\sin 2\alpha$, they simultaneously vanish at the points (7.4). The following solution cases of system (9.1) - (9.3) are possible:

1) $\alpha\neq\pi/2$, $\alpha\neq\pi$. Equations (9.1) - (9.3) have a zero solution $\varepsilon_{11}=\varepsilon_{12}=\varepsilon_{13}=\varepsilon_{22}=\varepsilon_{23}=\varepsilon_{33}=0$. From these equalities in the case of a linearly elastic body in the absence of a temperature load ($\Delta T=0$) all the stress tensor components also vanish at the special point $\sigma_{11}=\sigma_{12}=\sigma_{13}=\sigma_{22}=\sigma_{23}=\sigma_{33}=0$. At a temperature load, the solution for the stresses at the material point *A* will be $\sigma_{11}=\sigma_{22}=\sigma_{33}==-\dfrac{\omega\Delta T\,E}{1-2\nu}$, $\sigma_{12}=\sigma_{13}=\sigma_{23}=0$. In this case, the conditions on the surface of the body in the vicinity of the special point completely determine the stress state in it.

2) $\alpha=\pi/2$. From the equations (9.1) - (9.3) there are determined strains $\varepsilon_{22}=\varepsilon_{32}=\varepsilon_{33}=0$. Using these equations and physical equations, the dependencies between stresses at the edge points are obtained as: $\sigma_{33}-\nu\sigma_{11}-\nu\sigma_{22}=-\omega\Delta T\,E$; $\sigma_{22}-\nu\sigma_{11}-\nu\sigma_{33}=-\omega\Delta T\,E$; $\sigma_{32}=0$.

3) $\alpha=\pi$. Similarly to the previous case, strains are determined from equations (9.1) - (9.3) $\varepsilon_{11}=\varepsilon_{12}=\varepsilon_{22}=0$. With these equalities and physical equations the following

relationships between stresses may be obtained: $\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33} = -\omega \Delta T\, E;$
$\sigma_{22} - \nu\sigma_{11} - \nu\sigma_{33} = -\omega \Delta T\, E; \quad \sigma_{12} = 0.$

## 8. An edge with a rigidly clamped surface and a frictionless sliding surface

It is assumed that the forming surface 1 slides without friction along a rigid surface, and the forming surface 2 near the point $A$ is rigidly clamped. This means that tangent stresses at the edge points on an area oriented with the vector $\vec{n}$ vanish

$$\tau_{n'} = 0 \qquad \tau_{n2} = 0 \tag{10.1}$$

and on the area oriented with the vector $\overline{m}$, the relative elongations of the linear elements along the directions of the vectors $-\overline{m}'$, $\overline{r}_2$, and the shifts between these directions vanish

$$\varepsilon_{ij} m_i' m_j' = 0 \qquad \varepsilon_{ij} n_{2i} n_{2j} = 0 \qquad 2\varepsilon_{ij} m_i' n_{2j} = 0 \tag{10.2}$$

In terms of the stresses, the conditions (10.1), (10.2) are written by the equations

$$\begin{gathered} -\sigma_{11}\sin\alpha\cos\alpha - \sigma_{13}(\cos^2\alpha - \sin^2\alpha) + \sigma_{33}\sin\alpha\cos\alpha = 0 \\ \sigma_{11}(\cos^2\alpha - \nu\sin^2\alpha) + (1+\nu)\sigma_{13}\sin 2\alpha + \sigma_{33}(\sin^2\alpha - \nu\cos^2\alpha) - \nu\sigma_{22} = -\omega E\,\Delta T \\ \sigma_{11}\nu - \sigma_{33}\nu + \sigma_{22} = -\omega E\,\Delta T \end{gathered} \tag{10.3}$$

$$\begin{gathered} \sigma_{12}\sin\alpha + \sigma_{32}\cos\alpha = 0 \\ \sigma_{12}\cos\alpha + \sigma_{32}\sin\alpha = 0 \end{gathered} \tag{10.4}$$

Three Eqs. (10.3) form a system with four unknowns. The four possible determinants of the third rank of the matrix $\Delta_i (i = 1,2,3,4)$ are expressed by the equalities

$$\Delta_1 = -\nu(\nu+1)\cos^4 2\alpha$$
$$\Delta_2 = (\nu+1)[4\sin^4\alpha + (2\nu-5)\sin^2\alpha - (\nu-1)]$$
$$\Delta_3 = -0.5(2\nu^2 + \nu - 1)\sin 2\alpha$$
$$\Delta_4 = (\nu+1)[4\sin^4\alpha - (2\nu+3)\sin^2\alpha + \nu]$$

and for values $\nu$ in the interval $0 < \nu < 0.5$ do not simultaneously vanish. Consequently, equations (10.3) for all $\alpha$ are independent constraints on the stresses $\sigma_{11}, \sigma_{13}, \sigma_{33}, \sigma_{22}$ at a special point. The determinant of the matrix of system (10.4) $\Delta = -\cos 2\alpha$ in the domain of definition $\alpha$ vanishes at points

$$\alpha = \pi/4, \qquad \alpha = 3\pi/4. \tag{10.5}$$

Therefore, the following cases of solution behavior of Eqs. (10.3) and Eqs. (10.4) are possible:

1) $\alpha \neq \pi/4,\ \alpha \neq 3\pi/4$. Eqs. (10.4) have a solution $\sigma_{32} = 0,\ \sigma_{12} = 0$. The number of constraints on the components of the stress tensor at a special point is five.

2) If one of the values (10.5) for $\alpha$ is taken, the stresses $\sigma_{12}, \sigma_{32}$ become dependent

$$\sigma_{12} = -\sigma_{32}\ (\text{if } \alpha = \pi/4), \qquad \sigma_{12} = \sigma_{32}\ (\text{if } \alpha = 3\pi/4).$$

For both cases, the number of given independent constraints at the material point $A$ is greater than three. Further, the case of structural element deformation is considered when there are no displacements of the forming surface 1 in the direction of the vector $\overline{r}_2$. Such deformation is carried out, for example, in an axisymmetric body with its axisymmetric loading. In addition to the conditions (10.1), (10.2) the angle $2\alpha$ between the directions $\overline{n}', -\overline{m}'$ will be preserved (Fig. 1). Using formula (9.4) with the constancy of the angle for the case of a linearly thermoelastic body we may write as

$$\sigma_{11}[\cos^2\alpha(2-\cos 2\alpha)+\nu\sin^2\alpha(2+\cos 2\alpha)]+\sigma_{13}(1+\nu)\cos 2\alpha\sin 2\alpha - \\ -\sigma_{33}[\nu\cos^2\alpha(2-\cos 2\alpha)+\sin^2\alpha(2+\cos 2\alpha)]-\sigma_{22}\nu\cos 2\alpha = -\omega E\Delta T\cos 2\alpha \tag{10.6}$$

Adding this equality to equalities (10.3), we obtain a system of four linear inhomogeneous equations with four unknowns $\sigma_{11}, \sigma_{13}, \sigma_{33}, \sigma_{22}$. The determinant of the matrix of such a system

$$\Delta = -4(1-2\nu)(1+\nu)^2\sin^2\alpha(\sin^2\alpha-1)(\sin^2\alpha-0.5)$$

in the range of values $\alpha$ has the roots

$$\alpha_1 = \pi/4, \quad \alpha_2 = \pi/2, \quad \alpha_3 = 3\pi/4, \quad \alpha_4 = \pi. \tag{10.7}$$

Possible cases of the behavior of the solution of Eqs. (10.3), Eqs. (10.4) and Eqs. (10.6):

1) The angle $\alpha$ is not included in the list (10.7). The system of equations (10.3), (10.6) is consistent, the rank of its matrix is four. The only way is to determine the stresses $\sigma_{11}, \sigma_{13}, \sigma_{33}, \sigma_{22}$:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -\frac{\omega E\,\Delta T}{2\nu-1}, \quad \sigma_{13} = 0.$$

Eqs. (10.4) have a solution $\sigma_{32} = 0$, $\sigma_{12} = 0$. Six restrictions are superimposed on the stress tensor components at a special point.

2) $\alpha = \pi/4$. The rank of the matrix of the system (10.3), (10.6) and the rank of its extended matrix coincide and are equal to three. Independent restrictions are imposed on stress components

$$\sigma_{33} = \sigma_{11}, \sigma_{22} = 2\nu\sigma_{11} - \omega\Delta T\,E; \quad \sigma_{13} = (2\nu-1)\sigma_{11} - \omega\Delta T\,E.$$

Eq. (10.4) implies the dependence $\sigma_{12} = -\sigma_{32}$.

3) $\alpha = \pi/2$. The rank of the matrix of system (10.3), (10.6) and the rank of its extended matrix coincide. The stress components are related by the relations (taking into account the solution of Eqs. (10.4)):

$$\sigma_{33} = \sigma_{22} = \frac{1}{1-\nu}(\nu\sigma_{11} - \omega\Delta T\,E), \ \sigma_{13} = 0, \quad \sigma_{12} = 0, \ \sigma_{33} = 0.$$

In total there are five constraints.

4) $\alpha = 3\pi/4$. The rank of the matrix of the system (10.3), (10.6) and the rank of its extended matrix coincide and are equal to three. The stress components obey the dependences:

$$\sigma_{33} = \sigma_{11}, \ \sigma_{22} = 2\nu\sigma_{11} - \omega\Delta T\,E, \quad \sigma_{13} = -(2\nu-1)\sigma_{11} + \omega\Delta T\,E.$$

From the equations (10.4) we obtain $\sigma_{12} = \sigma_{32}$. There are in total four restrictions.

5) $\alpha = \pi$. Equations (10.3) and (10.6) are consistent, since the rank of the matrix of the system coincides with the rank of its extended matrix. Stress components obey restrictions

$$\sigma_{13} = 0, \quad \sigma_{33} = \frac{1-\nu}{\nu}\sigma_{11} + \frac{1}{\nu}\omega\Delta T\,E, \quad \sigma_{22} = \sigma_{11}.$$

Eqs. (10.4) imply $\sigma_{12} = \sigma_{32} = 0$. The number of restrictions at a special point exceeds the constraints number at the boundary points in the classical case.

## 9. Example: Streching of an element made up of two truncated cones

The axial section of the element is shown in Fig. 2. The problem is axisymmetric and is considered in a cylindrical coordinate system. The stress tensor components that are nonzero in this problem are, – $\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{zz}, \sigma_{rz}$. The solution is built for angles: 93º, 99º, 105º.

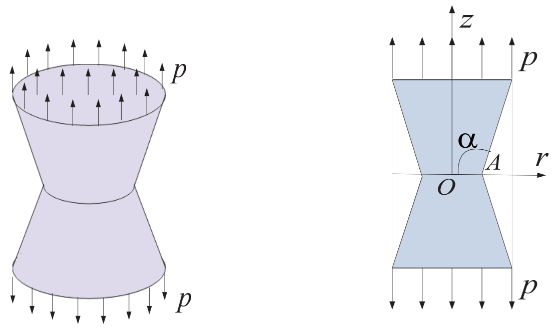


**Fig. 2.** An element composed of two truncated cones and its axial section

The material parameters have the following values: $E$ = 200GPa, $\nu = 0.3$, load $p$ = 100GPa. According to section 3 (case $\alpha \neq \pi/2, \alpha \neq \pi$), following stress components vanish at the material point $A$:

$$\sigma_{rr} = 0, \qquad \sigma_{zz} = 0, \qquad \sigma_{rz} = 0. \tag{11.1}$$

The number of constraints (11.1) exceeds the number of constraints (equal to two) at the usual surface point in the classical case, so the problem is non-classical. Its solution is constructed by the iterative numerical-analytic method (Pestrenin et al., 2017). For calculation, toroidal finite elements are adopted, the cross-section of which is an eight-node quadrilateral, which provides a quadratic approximation of the unknown functions. Directly adjacent to the node $A$ there are two elements with a characteristic size of 0.005mm. The solution in a small neighborhood of a special point is illustrated in Figs. 3, 4.

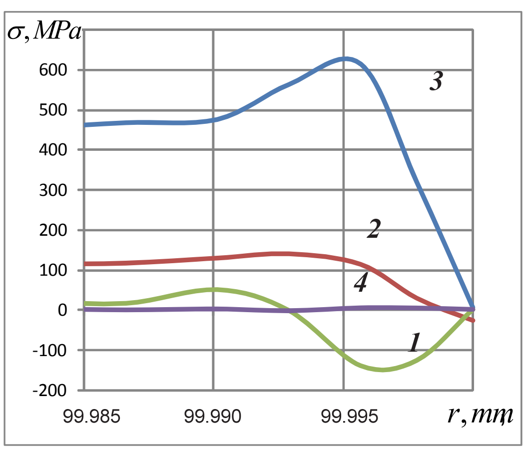


**Fig. 3.** Components of the stress tensor at the points of the axis $r$: 1–$\sigma_{rr}$; 2–$\sigma_{\varphi\varphi}$; 3–$\sigma_{zz}$; 4–$\sigma_{rz}$. $\alpha = 99°$.

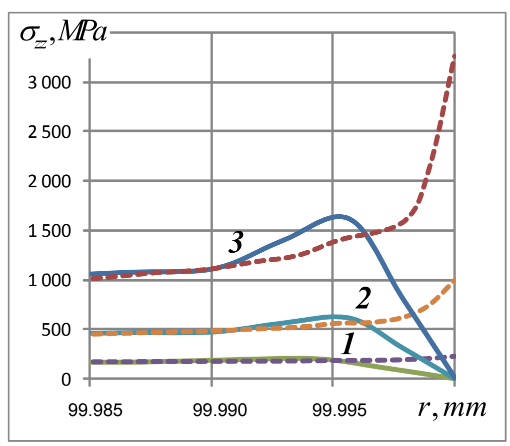


**Fig. 4.** Comparison of solutions $\sigma_{zz}$ for the iterative method (solid lines) and ANSYS-solutions (dashed lines) for the angle $\alpha$ : 1 – $93°$; 2 – $99°$; 3 – $105°$.

Fig. 3 shows dependence of the components of the stress tensor (in MPa) in the neighborhood of the point $A$ on $r$, when $\alpha = 99°$. It can be seen that only the circumferential stress does not vanish at the special point and all the conditions (11.1) are satisfied in it. Near the apex of the rib, the concentration of all stress components is observed, except for the shear stresses. In Fig. 4, in the same $r$-interval, the solution of the problem by the iterative method (solid lines) is compared with the classical solution obtained using the ANSYS engineering package (dashed lines). The stresses (in MPa) are shown in the vicinity of the point *A* for different values of the angle. It can be seen that the solutions differ only in a small neighborhood of the special point; solution in the classical formulation does not satisfy the conditions (11.1) and leads to overestimated stresses near the point *A*. Calculations were carried out on the TESLA-PGU supercomputer of the research and educational center for parallel and distributed computing at the Perm State National Research University (Russia).

## 10. Conclusions

On the basis of the concept of a continuous medium in the form of a continuum point and the elementary volume associated with it, generally accepted by researchers, an approach is proposed for constructing the specified restrictions at the edge points (special points) of an isotropic body. These restrictions are the state parameters of the elementary volume with the special point. The results of analyzing the stress-strain state of such volume are the following:

- preset restrictions at the points of the edge are formulated for various kinds of interactions between the edge-forming surfaces and the environment;
- it is shown that the number of preset restrictions at the points of the edge is usually redundant, which predetermines the non-classical formulation of solid mechanics problems for structural components with such features as 3D edges;
- critical combinations of material and geometric parameters are identified as well as load parameters corresponding to the singular stress state in elementary volumes with special points;
- restrictions at the load vector components are formulated to ensure the correct statements of mechanics problems for structural components with the considered special points.

The proposed approach makes it possible to study the stresses concentration in elementary volumes containing the spatial edge points of a deformable body and may be applied to mechanics problems of destruction and structural component strength studies.

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