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Determining the fabrication runtime for a buyer-vendor system with stochastic breakdown, accelerated rate, repairable items, and multi-delivery strategy

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C H R O N I C L E

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A B S T R A C T

This study explores the optimal fabrication runtime for a buyer-vendor incorporated system featuring repairable items, stochastic breakdown, accelerated rate, and multi-delivery strategy. Operating in today's competitive global market, transnational production firms make every effort to meet client requirements in terms of the due date and quality goods. Further, they also must handle all inevitable events incurred in the manufacturing process, such as unanticipated equipment breakdowns and defective products, with caution to avoid production schedule delay and cost overrun. To examine such a vendor-buyer incorporated system, we build a model to characterize the aforementioned features in the system. The function of total system cost is derived through formulation and analyses. The optimization method and a recursive algorithm are employed to help in deriving the optimal (i.e., cost minimization) fabrication runtime for our problem. An example numerically illustrates how our model, method, and algorithm work. It also reveals the capability of our model in analyzing the impact of each and/or joint feature(s) (e.g., the breakdown, accelerated rate, rework, multi-delivery strategy) on the system's utilization, optimal runtime, total expenses, and individual cost contributor to assist in managerial decision making, and hence, enabling the firm to gain competitive advantage.

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1. Introduction

This study explores the optimal fabrication runtime for a vendor-buyer incorporated system featuring repairable items, stochastic breakdown, accelerated rate, and multi-delivery strategy. Transnational manufacturers make every effort to meet client requirements in terms of due date and quality goods to retain their competitive advantage in the global market. The acceleration of the production process is often considered to meet orders' due dates, smooth production schedules, and balance equipment's loadings. Wijngaard (1979) examined a two-stage manufacturing system wherein buffer storage in-between stages of two unreliable fabrication units and each with different fabrication, breakdown, and repair rates. The author used regeneration points to analyze the impact on output for the aforementioned system. Distinct differential equations associated with numerical illustrations were applied to handle different assumptions on fabrication rates. Moon et al. (1991) investigated the influence of controllable manufacturing rates on a multiproduct single-machine rotation-schedule fabrication system. The authors studied the impact of changing manufacturing rates during production runs on the optimal policy/cost of

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the system and numerically illustrated their points and results. Sethi et al. (2000) used the verification theorem to develop an optimal control on a single-machine two-part type deterministic product system. The authors found that under certain conditions, the unique optimal control could be obtained and they also extended the model to explore the two-part type single-machine unreliable stochastic system. Sharma (2011) studied the effect of the production rate increase on the product quality in the manufacturer-supplier environments. Formulation was used, analyzed, and discussed among various cases to help in gaining managerial decisional information. AlDurgam et al. (2017) investigated a stochastic inventory model wherein a manufacturer uses various fabrication rates to cope with uncertain demand and reduce lead times, and distribution of the finished goods in multi-truckload was considered. Minimization of the total expenses of production and distribution was intended. The authors illustrated their model numerically and discussed the potential benefits that can be gained in the coordinated supply chain members. Other studies (Arslan et al., 2001; Gharbi et al., 2008; Chiu et al., 2018, 2019a) explored the influence of diverse characteristics of accelerated manufacturing rates on the production systems and supply-chain environments.

The fixed time-interval multi-delivery strategy of end products is regularly implemented by the deterministic inventory systems in buyer-vendor incorporated environments. Banerjee and Banerjee (1994) examined a single-product single-vendor multi-buyer stochastic inventory model, wherein the analytical approaches employed including a common-cycle based and an electronic data interchange. The authors showed that by using an algorithm they proposed to compute the system's operating variables, the results derived from their model can facilitate decision makings for all parties in the supply-chain system. Bylka (2003) built a model based on various game scenarios on shipment frequencies to explore the competitive and cooperative policies for a dynamic decentralized batch fabrication-distribution system. The research results were discussed and compared to that in the existing literature. Hemmati et al. (2016) examined a multi-item short sea stock-routing model, wherein a many-to-many continuous-time delivery structure under a mixed fleet of ships framework is considered. The authors employed a specific two-phase hybrid matheuristic to resolve their problem. In phase one, their model was converted into the ship scheduling and routing model and solved by mathematical programming. In phase two, the neighborhood searching approach was used to further resolve the result of phase one. Finally, for small-size cases, their heuristic was compared with the exact algorithm, then their proposed model was applied to larger/realistic instances. Other works (Díaz-Mateus et al., 2018; Urrea and Pascal, 2018; Chiu et al., 2019b; Kumar et al., 2019; Batukhtin et al., 2019; Yadavalli et al., 2019) studied the influence of diverse characteristics of multi-delivery strategy on different fabrication-shipment systems in supply-chain environments.

The inevitable events, such as unanticipated equipment breakdowns and defective products, may randomly incur in the manufacturing process and cause production schedule delay and cost overrun. Thus, such events and their consequent effects/actions have brought extensive attention to production practitioners and researchers in past decades. Gershwin and Berman (1981) examined a transfer line comprising two unreliable equipment and a storage buffer, wherein the exponential service/failure and repair rates are associated with these machines. The authors employed the Markov model to analyze the parts' movement and process in the system. Certain system's behavior was numerically analyzed and illustrated. Gunasekaran et al. (1991) used mathematical modeling to explore a multi-stage, multiproduct inventory replenishment system, wherein multiple production facilities are subject to random failures. Their study aimed to investigate the impact of facility unreliability on the optimal batch size and total expenses of the system. The authors use an example to explain the system's applicability and behavior. Gong and Matsuo (1997) explored a multi-stage multiproduct fabrication system with uncertain yield, rework, and random demand. The authors aimed to develop an optimal policy to not only satisfy demand but also minimize the level of work-in-process. The authors presented proof of the optimal policy for their stochastic dynamic model. Lin and Kroll (2006) studied an imperfect EMQ model with random breakdowns. Two exponential-distribution variables were assumed in their model, namely, the times to breakdown and to shift. Their aim was to find the cost-minimization batch size for the model. Moussawi-

Haidar et al. (2016) incorporated the screening time and rework process into the classic economic production model, and examined two scenarios for handling the imperfect quality products, namely (i) reworking them and (ii) selling them at a discount. They aimed to maximize the expected profit for the system. The renewal reward theory was used and the optimal fabrication batch size was determined. An example was used to numerically illustrate how their model work, including the deriving the lot-size solution and performing sensitivity analyses to diverse system parameters. Other studies (Ghalme et al., 2017; Zhao et al., 2018; Okonkwo et al., 2018; Rao and Singh, 2018; Iqbal et al., 2019; Dan-asabe et al., 2019; Wenbin et al., 2019; Zahedi et al., 2019) investigated the impact of diverse aspects of unreliable production equipment and imperfect quality items produced on various manufacturing systems and production planning. As prior works paid little attention to the joint influence of breakdown, accelerated rate, repairable items, and multi-delivery strategy on the fabrication runtime decision in buyer-vendor environments, this research aims to fill this gap.

2. The proposed problem

2.1. Notation

Q = the replenishment batch size,
λ = annual demand,
t_{1A} = the machine uptime in the breakdown occurring case – decision variable,
t'_{2A} = the rework time in the breakdown occurring case,
t'_{3A} = delivery time of finished stocks in the breakdown occurring case,
T'_A = cycle time in the breakdown occurring case,
t = time to a stochastic breakdown occurs,
M = breakdown repair cost,
β = mean Poisson distributed breakdown rate,
t_r = breakdown repair time,
P_{1A} = accelerated annual manufacturing rate,
P_1 = normal annual manufacturing rate,
α_1 = the accelerated proportion as compared with P_1,
C_A = unit cost when accelerated rate P_{1A} is used,
C = unit cost when normal rate P_1 is used,
K_A = setup cost when accelerated rate P_{1A} is used,
K = setup cost when normal rate P_1 is used,
α_2 = setup cost increase proportion when accelerated rate P_{1A} is used,
P_{2A} = accelerated annual reworking rate,
P_2 = normal annual reworking rate,
C_{RA} = unit reworking cost when accelerated rate P_{2A} is used,
C_R = unit reworking cost when normal rate P_2 is used,
α_3 = unit cost increase proportion when accelerated rate P_{1A} and P_{2A} are used,
t'_{nA} = fixed time interval between two consecutive deliveries in the breakdown occurring case,
C_T = unit delivery cost,
K_1 = fixed delivery cost,
n = equal-quantity deliveries per cycle,
D = the number of products per delivery,
I = leftover quantities at the end of each delivery time interval,
h = unit holding cost,
h_1 = reworked product's unit holding cost,
h_2 = buyer stock's unit holding cost,
x = random nonconforming proportion during the manufacturing process,
d_{1A} = output rate of nonconforming stocks in uptime t_{1A} when accelerated rate P_{1A} is used,
d_1 = output rate of nonconforming stocks in t_{1A} when normal rate P_1 is used,

C_1 = safety stock's unit cost,
h_3 = safety stock's unit holding cost,
H_0 = level of end products when a breakdown occurs,
H_1 = level of finished products when uptime t_{1A} ends,
H = level of finished products when rework time t_{2A} ends,
g = t_r, breakdown repair time,
t_{2A} = reworking time in the no breakdown occurring case,
t_{3A} = delivery time in the no breakdown occurring case,
T_A = cycle time in the no breakdown occurring case,
t_{nA} = fixed time interval between two consecutive deliveries in the no breakdown occurring case,
$I(t)$ = level of perfect products at time t,
$I_c(t)$= level of buyer's products at time t,
$I_F(t)$ = level of safety products at time t,
$I_d(t)$= level of nonconforming products at time t,
t_1 = uptime for a system without breakdown, nor accelerated rate,
t_2 = reworking time for a system without breakdown, nor accelerated rate,
t_3 = delivery time for a system without breakdown, nor accelerated rate,
T = cycle time for a system without breakdown, nor accelerated rate,
$\boldsymbol{T}_{\mathbf{A}}$ = cycle time for the proposed system with/without a breakdown occurring,
$E[\boldsymbol{T}_{\mathbf{A}}]$ = expected cycle time length for the proposed system,
$TC(t_{1A})_1$ = total cost per cycle in the breakdown occurring case,
$E[TC(t_{1A})_1]$ = expected total cost per cycle in the breakdown occurring case,
$TC(t_{1A})_2$ = total cost per cycle in the no breakdown occurring case,
$E[TC(t_{1A})_2]$ = expected total cost per cycle in the no breakdown occurring case,
$E[TCU(t_{1A})]$ = the expected system cost per unit time for the proposed problem, whether or not a breakdown occurs.

2.2. *Description and modeling*

This study considers the fabrication runtime problem for a vendor-buyer system with the stochastic breakdown, accelerated rate, repairable items, and multi-delivery rule. In the batch fabrication planning stage, to reduce cycle time or to smooth overall fabrication schedules, considering the option of an accelerated fabrication rate to meet annual demand λ could be an effective approach. Subsequently, a higher unit C_A and setup K_A costs are accompanied by such an extra speedup proportion α_1 in the accelerated rate P_{1A}. The following are the correlations of parameters P_{1A}, C_A, and K_A and their relevant normal-rate variables:

$$P_{1A} = (1+\alpha_1)P_1 \quad (1)$$

$$C_A = (1+\alpha_3)C \quad (2)$$

$$K_A = (1+\alpha_2)K \quad (3)$$

Also, in real manufacturing settings, due to unforeseen issues a random x proportion of nonconforming products can be manufactured at a rate d_{1A} (i.e., xP_{1A}). No stock-out conditions are allowed, thus, we assume $P_{1A} - d_{1A} - \lambda > 0$. All nonconforming are assumed to be repairable at an accelerated rate P_{2A} and unit rework cost C_{RA}, right after the end of uptime in each batch cycle, where

$$C_{RA} = (1+\alpha_3)C_R \quad (4)$$

$$P_{2A} = (1+\alpha_1)P_2 \quad (5)$$

Furthermore, the fabrication equipment is subject to a Poisson distributed breakdown rate β. When a breakdown occurs, the equipment is promptly under repair, and once the repair job is done, the fabrication of the interrupted batch resumes immediately. The repair time t_r is assumed constant, however, if actual time spending on repair exceeds t_r, then, rental equipment is in place to prevent further schedule delay. Upon accomplishment of batch production, n equal-quantity shipments are transported to the buyer at fixed time interval t'_{nA} during delivery time t'_{3A}. Due to the nature of stochastic breakdown, the following two separate conditions are studied:

2.2.1. Condition 1: A stochastic breakdown occurs in t_{1A}

The level of perfect products in condition 1 (i.e., when $t < t_{1A}$) is depicted in Fig. 1. It specifies that when a breakdown occurs, the stock level accumulates to H_0; after the equipment repair task is accomplished, the stock level continues to rise to H_1 when t_{1A} ends; and when the rework is completed, the stock level ends at H. Then, the delivery of multiple shipments begins.

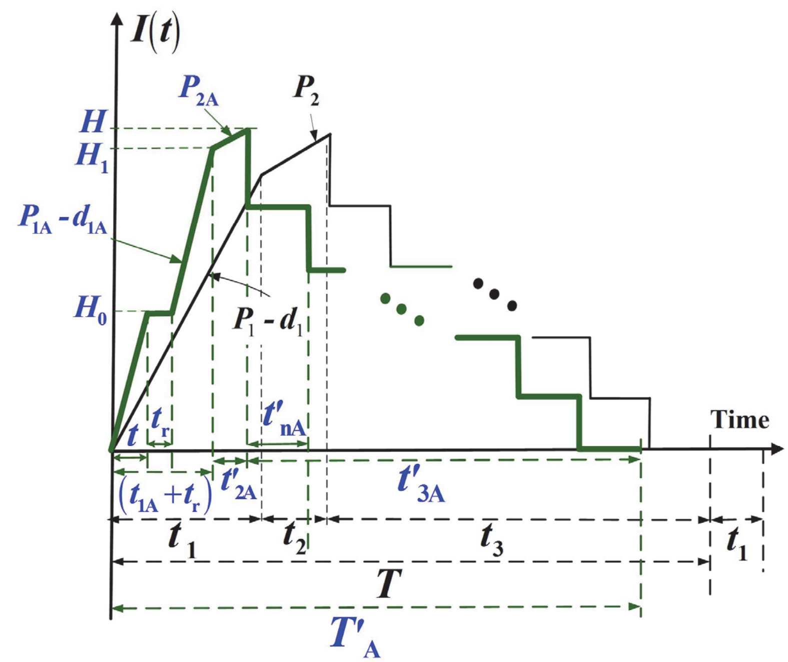

Fig. 1. The level of perfect products in condition 1 of the proposed problem (in green) as compared to that of the same problem without breakdown, nor accelerated rate option (in black)

The level of safety items in condition 1 of the proposed problem is displayed in Fig. 2. It specifies that safety items λt_r are used to meet the demand during repair time, thus, they must be delivered along with the finished lot in t'_{3A}.

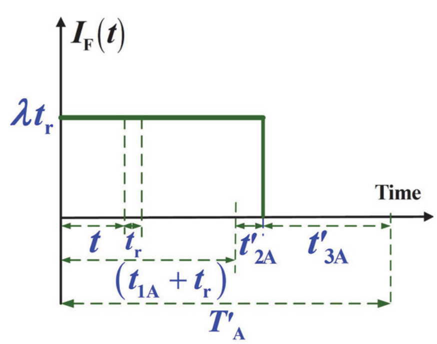

Fig. 2. The level of safety items in condition 1 of the proposed problem

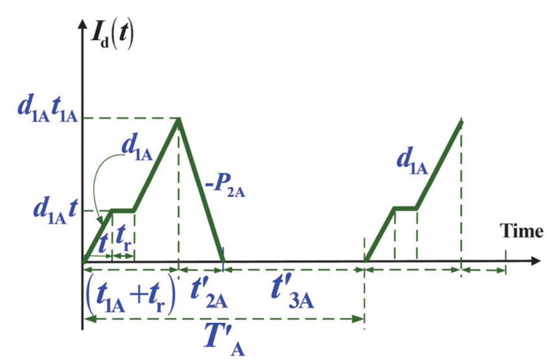

Fig. 3. The level of repairable nonconforming products in condition 1 of the proposed problem

The level of repairable defective items in condition 1 of the proposed problem is exhibited in Fig. 3. It indicates that when a breakdown occurs, the defective items accumulate to $d_{1A}t$; at the end of t_{1A} it comes up to $d_{1A}t_{1A}$, and at the end of rework time t'_{2A} it depletes to zero. According to the aforementioned description along with observation from Fig. 1 to Fig. 3, we obtain the following formulas:

$$H_0 = (P_{1A} - d_{1A})t \tag{6}$$

$$H_1 = (P_{1A} - d_{1A})t_{1A} \tag{7}$$

$$t_{1A} = \frac{Q}{P_{1A}} = \frac{H_1}{P_{1A} - d_{1A}} \tag{8}$$

$$t'_{2A} = \frac{xQ}{P_{2A}} \tag{9}$$

$$T'_A = t_{1A} + t'_{2A} + t'_{3A} + t_r = \frac{Q}{\lambda} + t_r \tag{10}$$

$$t'_{3A} = T'_A - (t_{1A} + t'_{2A} + t_r) = \frac{Q}{\lambda}\left(1 - \frac{\lambda}{P_{1A}} - \frac{\lambda x}{P_{2A}}\right) \tag{11}$$

$$H = H_1 + (P_{2A} - d_{2A})t'_{2A} + \lambda t_r = Q + \lambda t_r \tag{12}$$

$$d_{1A}t_{1A} = (xP_{1A})t_{1A} = xQ. \tag{13}$$

The level of finished products during the delivery time in condition 1 is exhibited in Fig. 4. The total inventories in t'_{3A} (Chiu et al., 2019b) is shown in Eq. (14).

$$\left(\frac{1}{n^2}\right)\left(\sum_{i=1}^{n-1} i\right)H(t'_{3A}) = \left(\frac{n-1}{2n}\right)H(t'_{3A}) \tag{14}$$

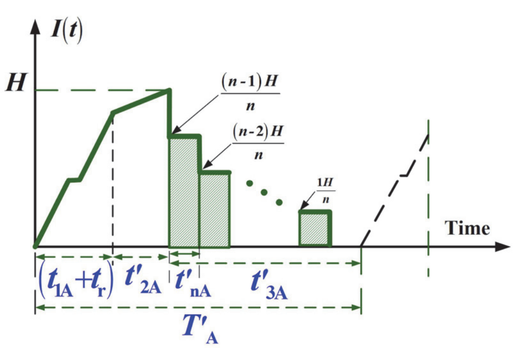

Fig. 4. The level of finished products during the delivery time in condition 1 of the problem

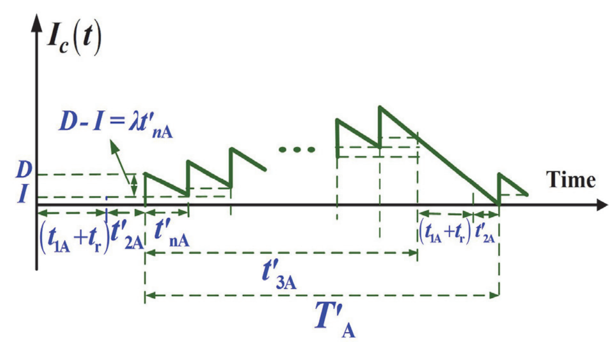

Fig. 5. Buyer's inventory status in a replenishment cycle in condition 1 of the problem

Buyer's inventories in a replenishment cycle in condition 1 of the proposed problem is revealed in Fig. 5. The total inventories in T'_A can be obtained (Chiu et al., 2019b) as follows:

$$\frac{n(n-1)}{2}I(t'_{nA}) + n\left(D - \frac{\lambda(t'_{nA})}{2}\right)(t'_{nA}) + \frac{nI}{2}(t_{1A} + t'_{2A}) = \frac{1}{2}\left[(H - \lambda t'_{3A})T'_A + \frac{Ht'_{3A}}{n}\right] \tag{15}$$

Where

$$I = D - (\lambda t'_{nA}) \tag{16}$$

$$D = \frac{H}{n}. \tag{17}$$

Total cost per cycle in condition 1, $TC(t_{1A})_1$ includes the variable and fixed accelerated fabrication costs, breakdown repair cost, safety products related cost, rework cost, fixed and variable delivery costs, and overall holding costs in T'_A (including reworked stocks, buyer's stocks, and the perfect and nonconforming stocks in the entire cycle) as follows:

$$\begin{aligned} TC(t_{1A})_1 &= C_A Q + K_A + M + C_1(\lambda t_r) + \left[h_3(\lambda t_r)(t_{1A} + t_r + t'_{2A})\right] + C_{RA} x Q \\ &+ nK_1 + C_T\left[Q + \lambda t_r\right] + h_1 \frac{P_{2A} t'_{2A}}{2}(t'_{2A}) + \frac{h_2}{2}\left[(H - \lambda t'_{3A}) T'_A + \frac{H t'_{3A}}{n}\right] \\ &+ h\left[\frac{H_1 + d_{1A} t_{1A}}{2}(t_{1A}) + (H_0 t_r) + (d_{1A} t) t_r + \frac{H_1 + (H - \lambda t_r)}{2}(t'_{2A}) + \left(\frac{n-1}{2n}\right) H t'_{3A}\right] \end{aligned} \tag{18}$$

We employ the expected values of x to deal with its randomness and substitute Eqs. (1) to (17) in Eq. (18), the following $E[TC(t_{1A})_1]$ is obtained:

$$\begin{aligned} E\left[TC(t_{1A})_1\right] &= \left[(1+\alpha_2)K\right] + \left[(1+\alpha_3)C\right]\left[(1+\alpha_1)P_1 t_{1A}\right] + C_1\lambda g + h_3 g\left[\left[(1+\alpha_1)P_1 t_{1A}\right]y_1 + \lambda g\right] \\ &+ M + nK_1 + C_T\left[\left[(1+\alpha_1)P_1 t_{1A}\right] + \lambda g\right] + (1+\alpha_3)C_R E[x]\left[(1+\alpha_1)P_1 t_{1A}\right] \\ &+ \frac{E[x]^2\left[(1+\alpha_1)P_1 t_{1A}\right]^2(h_1 - h)}{2\left[(1+\alpha_1)P_2\right]} + \frac{\left[(1+\alpha_1)P_1 t_{1A}\right]^2(h_2 - h)(1-y_1)}{2n\lambda} + \frac{h_2\left[(1+\alpha_1)P_1 t_{1A}\right]^2}{2}\left(\frac{y_1}{\lambda}\right) \\ &+ \frac{h\left[(1+\alpha_1)P_1 t_{1A}\right]^2}{2\lambda}\left[1 + \frac{\lambda E[x]}{(1+\alpha_1)P_2}\right] + hg\left[\left[(1+\alpha_1)P_1\right]t + \frac{(1+\alpha_1)P_1 t_{1A}}{2}(1-y_1)\right] \\ &+ \frac{(h_2 - h)g}{2n}\left[(1+\alpha_1)P_1 t_{1A}\right](1-y_1) + \frac{h_2 g}{2}\left[\left[(1+\alpha_1)P_1 t_{1A}\right](1+y_1) + \lambda g\right] \end{aligned} \tag{19}$$

where

$$y_1 = \frac{\lambda}{(1+\alpha_1)}\left[\frac{1}{P_1} + \frac{E[x]}{P_2}\right].$$

2.2.2. Condition 2: No stochastic breakdowns occur in t_{1A}

The level of perfect products in condition 2 (i.e., when $t \geq t_{1A}$) is displayed in Fig. 6. It specifies that the stock level accumulates to H_1 at the end of uptime, and it reaches H at the end of t_{2A}. Then, the delivery of multiple shipments begins.

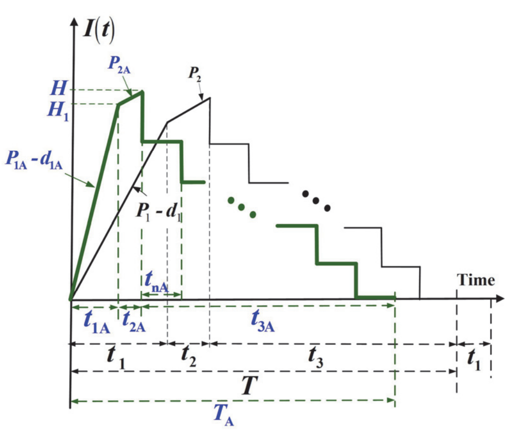

Fig. 6. The level of perfect products in condition 2 of the proposed problem (in green) as compared to that of the same problem without accelerated rate option (in black)

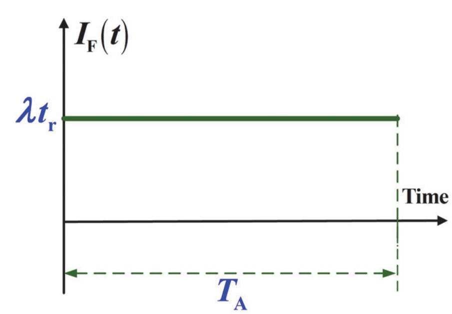

Fig. 7. The level of safety items in condition 2 of the proposed problem

The level of safety items in condition 2 of the proposed problem is displayed in Fig. 7. It specifies that during the entire cycle time the safety items λt_r were not used because no breakdowns occurred. The status of defective products comes up to $d_{1A}t_{1A}$ at the end of uptime t_{1A}, and it depletes to zero when rework time t_{2A} ends (refer to Fig. 3 excepts that no breakdown occurs and the following parameters are used: t_{2A}, t_{3A}, and T_A). According to the description of condition 2, along with observation from Fig. 6 to Fig. 7, we obtain the following formulas:

$$H_1 = \left(P_{1A} - d_{1A}\right)t_{1A} \tag{20}$$

$$H = H_1 + \left(P_{2A} - d_{2A}\right)t_{2A} \tag{21}$$

$$T_A = t_{1A} + t_{2A} + t_{3A} = \frac{Q}{\lambda} \tag{22}$$

$$t_{1A} = \frac{Q}{P_{1A}} = \frac{H_1}{P_{1A} - d_{1A}} \tag{23}$$

$$t_{2A} = \frac{xQ}{P_{2A}} \tag{24}$$

$$t_{3A} = T_A - \left(t_{1A} + t_{2A}\right) = \frac{Q}{\lambda}\left[1 - \frac{\lambda}{P_{1A}} - \frac{\lambda x}{P_{2A}}\right]. \tag{25}$$

For the level of perfect products in delivery time in condition 2, we can refer to Fig. 4 but replacing the following variables: t_{2A}, t_{nA}, t_{3A}, and T_A in it. The total inventories in t_{3A} (Chiu et al., 2019b) is shown as follows:

$$\left(\frac{1}{n^2}\right)\left(\sum_{i=1}^{n-1} i\right)H\left(t_{3A}\right) = \left(\frac{n-1}{2n}\right)H\left(t_{3A}\right) \tag{26}$$

For the level of buyer's inventories in a replenishment cycle in condition 2, we can refer to Fig. 5 but without having t_r and replacing the following variables: t_{2A}, t_{nA}, t_{3A}, and T_A in it. The total inventories in T_A (Chiu et al., 2019b) are as follows:

$$\frac{1}{2}\left[\left(H - \lambda t_{3A}\right)T_A + \frac{Ht_{3A}}{n}\right]. \tag{27}$$

Therefore, $TC(t_{1A})_2$ includes the variable and fixed accelerated fabrication costs, safety products holding cost, rework cost, fixed and variable delivery costs, and overall holding costs in T'_A (including reworked stocks, buyer's stocks, and the perfect and nonconforming stocks in the entire cycle) as follows:

$$\begin{aligned} TC\left(t_{1A}\right)_2 &= C_A Q + K_A + h_3\left(\lambda t_r\right)T_A + C_{RA}xQ + \frac{h_2}{2}\left[\left(H - \lambda t_{3A}\right)T_A + \frac{Ht_{3A}}{n}\right] + nK_1 \\ &\quad + C_T Q + h_1\frac{P_{2A}t_{2A}}{2}\left(t_{2A}\right) + h\left[\frac{H_1 + d_{1A}t_{1A}}{2}\left(t_{1A}\right) + \frac{H_1 + H}{2}\left(t_{2A}\right) + \left(\frac{n-1}{2n}\right)Ht_{3A}\right] \end{aligned} \tag{28}$$

We use the expected values of x to deal with its randomness and substitute Eq. (13), and Eq. (20) to Eq. (27) in Eq. (28), the following $E[TC(t_{1A})_2]$ is gained:

$$\begin{aligned} E\left[TC\left(t_{1A}\right)_2\right] &= \left[\left(1+\alpha_2\right)K\right] + \left[\left(1+\alpha_3\right)C\right]\left[\left(1+\alpha_1\right)P_1t_{1A}\right] + nK_1 + C_T\left[\left(1+\alpha_1\right)P_1t_{1A}\right] \\ &\quad + \left(1+\alpha_3\right)C_R E[x]\left[\left(1+\alpha_1\right)P_1t_{1A}\right] + h_3 g\left[\left(1+\alpha_1\right)P_1t_{1A}\right] \\ &\quad + \frac{E[x]^2\left[\left(1+\alpha_1\right)P_1t_{1A}\right]^2\left(h_1 - h\right)}{2\left[\left(1+\alpha_1\right)P_2\right]} + \frac{\left[\left(1+\alpha_1\right)P_1t_{1A}\right]^2\left(h_2 - h\right)\left(1 - y_1\right)}{2n\lambda} \\ &\quad + \frac{h_2\left[\left(1+\alpha_1\right)P_1t_{1A}\right]^2}{2}\left(\frac{y_1}{\lambda}\right) + \frac{h\left[\left(1+\alpha_1\right)P_1t_{1A}\right]^2}{2\lambda}\left[1 + \frac{\lambda E[x]}{\left(1+\alpha_1\right)P_2}\right] \end{aligned} \tag{29}$$

where

$$y_1 = \frac{\lambda}{(1+\alpha_1)}\left[\frac{1}{P_1}+\frac{E[x]}{P_2}\right].$$

3. Resolving the proposed problem

Because of the Poisson distributed breakdown rate has mean = β, thus, the time to a breakdown occurrence follows the Exponential distribution with density function $\beta e^{-\beta t_{1A}}$ (i.e., $f(t)$) and cumulative density function $(1 - e^{-\beta t_{1A}})$ (i.e., F(t)). Therefore, we can calculate $E[TCU(t_{1A})]$ from Eq. (30).

$$E\left[TCU\left(t_{1A}\right)\right]=\frac{\left\{\int_0^{t_{1A}} E\left[TC\left(t_{1A}\right)_1\right]\cdot f\left(t\right)dt+\int_{t_{1A}}^{\infty} E\left[TC\left(t_{1A}\right)_2\right]\cdot f\left(t\right)dt\right\}}{E[\boldsymbol{T}_A]} \tag{30}$$

where $E[\boldsymbol{T}_A]$, T'_A, and T_A represent the following:

$$E[\boldsymbol{T}_A]=\int_0^{t_{1A}} T'_A \cdot f(t)dt+\int_{t_{1A}}^{\infty} T_A\cdot f(t)dt \tag{31}$$

$$T'_A=\frac{Q+\lambda t_r}{\lambda}=\frac{t_{1A}P_{1A}+\lambda t_r}{\lambda} \tag{32}$$

$$T_A=\frac{Q}{\lambda}=\frac{t_{1A}P_{1A}}{\lambda} \tag{33}$$

Substitute Eq. (19), Eq. (29), and Eq. (31) in Eq. (30), together with additional derivation efforts, we gain the following $E[TCU(t_{1A})]$ (Appendix A shows the details):

$$E\left[TCU\left(t_{1A}\right)\right]=\left[\frac{\lambda}{1+\frac{\lambda g\left(1-e^{-\beta t_{1A}}\right)}{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}}\right]\left[\frac{v_0}{t_{1A}}+\frac{v_1}{t_{1A}}+v_2e^{-\beta t_{1A}}+\frac{v_3e^{-\beta t_{1A}}}{t_{1A}}-v_4e^{-\beta t_{1A}}+v_4+v_5t_{1A}+v_6\right] \tag{34}$$

Apply the first- and second-derivatives of $E[TCU(t_{1A})]$ we obtain Eqs. (B-1) and (B-2) (see Appendix B). When Eq. (B-3) holds, we can solve t_{1A}* by setting Eq. (B-1) equal to zero. Since the first term on the RHS of Eq. (B-1) is positive, we obtain the following:

$$\begin{bmatrix}-\left(v_0+v_1\right)\left[\left(1+\alpha_1\right)P_1+e^{-\beta t_{1A}}\lambda g\beta\right]-\left(v_4+v_6\right)\left(\lambda g\right)\left(e^{-\beta t_{1A}}\beta\left(t_{1A}\right)+e^{-\beta t_{1A}}-1\right)\\+\left(v_2-v_4\right)\left[-e^{-\beta t_{1A}}\beta\left(t_{1A}\right)^2\left(1+\alpha_1\right)P_1-e^{-\beta t_{1A}}\beta\left(t_{1A}\right)\lambda g-e^{-2\beta t_{1A}}\lambda g+e^{-\beta t_{1A}}\lambda g\right]\\+v_3\left[-e^{-\beta t_{1A}}\beta\left(t_{1A}\right)\left(1+\alpha_1\right)P_1-e^{-\beta t_{1A}}\beta\lambda g-e^{-\beta t_{1A}}\left(1+\alpha_1\right)P_1\right]\\+v_5\left(t_{1A}\right)\left[\left(t_{1A}\right)\left(1+\alpha_1\right)P_1+2\lambda g\left(1-e^{-\beta t_{1A}}\right)-e^{-\beta t_{1A}}\left(t_{1A}\right)\lambda g\beta\right]\end{bmatrix}=0 \tag{35}$$

Let w_0, w_1, and w_2 denote the following:

$$w_0=\left[\left(v_2-v_4\right)\left[-e^{-\beta t_{1A}}\beta\left(1+\alpha_1\right)P_1\right]+v_5\left[\left(1+\alpha_1\right)P_1-e^{-\beta t_{1A}}\lambda g\beta\right]\right]$$

$$w_1=-\left(e^{-\beta t_{1A}}\beta\right)\left[\left(v_4+v_6\right)\left(\lambda g\right)+\left(v_2-v_4\right)\left(\lambda g\right)+v_3\left(1+\alpha_1\right)P_1\right]+v_5\left[2\lambda g\left(1-e^{-\beta t_{1A}}\right)\right]$$

$$w_2=\begin{bmatrix}-\left(v_0+v_1\right)\left[\left(1+\alpha_1\right)P_1+e^{-\beta t_{1A}}\lambda g\beta\right]-\left(v_4+v_6\right)\left(\lambda g\right)\left(e^{-\beta t_{1A}}-1\right)\\+\left(v_2-v_4\right)\lambda g\left(-e^{-2\beta t_{1A}}+e^{-\beta t_{1A}}\right)-v_3e^{-\beta t_{1A}}\left[\beta\lambda g+\left(1+\alpha_1\right)P_1\right]\end{bmatrix}$$

Eq. (35) is rearranged as follows:

$$w_0\left(t_{1A}\right)^2+w_1\left(t_{1A}\right)+w_2=0 \tag{36}$$

Apply the square root solution, the following t_A* is determined:

$$t_{1A}{}^*=\frac{-w_1\pm\sqrt{w_1^2-4w_0w_2}}{2w_0} \tag{37}$$

Since $F(t_{1A}) = (1 - e^{-\beta t1A})$ is over the range [0, 1], so does its complement $e^{-\beta t1A}$. Also, Eq. (35) can be further rearranged as follows:

$$e^{-\beta t_{1A}}=\frac{-\left(v_0+v_1\right)\left(1+\alpha_1\right)P_1+\left(v_4+v_6\right)\left(\lambda g\right)+v_5t_{1A}\left[t_{1A}\left(1+\alpha_1\right)P_1+2\lambda g\right]}{\left\{\begin{matrix}\left(v_0+v_1\right)\lambda g\beta+\left(v_4+v_6\right)\lambda g\left(\beta t_{1A}+1\right)+v_3\left[\beta\lambda g+\left(1+\alpha_1\right)P_1\left(1+\beta t_{1A}\right)\right]\\+\left(v_2-v_4\right)\left[\beta t_{1A}^2\left(1+\alpha_1\right)P_1+\lambda g\left(\beta t_{1A}+e^{-\beta t_{1A}}-1\right)\right]+v_5\lambda g t_{1A}\left(2+t_{1A}\beta\right)\end{matrix}\right\}} \tag{38}$$

We start with the bound of $e^{-\beta t1A}$ to search for the optimal t_{1A}*. Initially, apply Eq. (37) using $e^{-\beta t1A} = 0$ and $e^{-\beta t1A} = 1$ to obtain the initial bound for t_{1A} (i.e., t_{1AU} and t_{1AL}). Next, using the current t_{1AU} and t_{1AL} to gain the update values of $e^{-\beta t1AU}$ and $e^{-\beta t1AL}$. Then, re-apply Eq. (37) using the current $e^{-\beta t1AU}$ and $e^{-\beta t1AL}$ to obtain a set of update bounds t_{1AU} and t_{1AL}. If $t_{1AU} = t_{1AL}$, then t_{1A}* is derived (i.e., $t_{1A}^* = t_{1AU} = t_{1AL}$); otherwise, repeat the aforementioned iterations until $t_{1AU} = t_{1AL}$.

4. Numerical demonstration

A numerical example with the following parameters (see Table 1) is offered to demonstrate the research result's applicability.

Table 1

The parameters used in this numerical demonstration

Parameters	P_{1A}	C_A	P_{2A}	K_A	C_{RA}	K_1	M	C_1	λ	h	β	α_1	α_2
Values	15000	2.5	7500	220	1.25	90	2500	2.0	4000	0.4	1	0.5	0.1
	P_1	C	P_2	K	C_R	n	g	C_T	x	h_1	h_2	h_3	α_3
	10000	2.0	5000	200	1.0	3	0.018	0.01	20%	0.4	1.6	0.4	0.25

The prerequisite of solving the proposed problem is to ensure the convexity of $E[TCU(t_{1A})]$, that is $\delta(t_{1A}) > t_{1A} > 0$ (or Eq. (B-3) holds). First, let $e^{-\beta t1A} = 0$ and $e^{-\beta t1A} = 1$, we apply Eq. (37) with $\beta = 1.0$ to obtain $t_{1AU} = 0.2953$ and $t_{1AL} = 0.0881$, thus find $e^{-\beta t1AU} = 0.7443$ and $e^{-\beta t1AL} = 0.9157$. Then, apply Eq. (B-3) with $e^{-\beta t1AU}$ and $e^{-\beta t1AL}$, respectively, so convexity of $E[TCU(t_{1A})]$ for $\beta = 1$ is confirmed, since $\delta(t_{1AU}) = 0.5205 > t_{1AU} = 0.2953 > 0$ and $\delta(t_{1AL}) = 0.2886 > t_{1AL} = 0.0881 > 0$. Table C exhibits the outcomes of more convexity tests with a wider choice of β values to simply demonstrate the applicability of our study.

Once we confirmed the cost function is convex, a recursive algorithm proposed at the end of previous subsection can be employed to locate t_{1A}* as follows: First, apply Eq. (37) with $e^{-\beta t1A} = 0$ and $e^{-\beta t1A} = 1$ to gain the initial values of $t_{1AU} = 0.2953$ and $t_{1AL} = 0.0881$, then use them to obtain $e^{-\beta t1AU} = 0.7443$ and $e^{-\beta t1AL} = 0.9157$. Repeatedly apply Eq. (37) with current $e^{-\beta t1AU}$ and $e^{-\beta t1AL}$ to calculate/update t_{1AU} and t_{1AL}, and compute/update the values of $e^{-\beta t1AU}$ and $e^{-\beta t1AL}$ until $t_{1AU} = t_{1AL}$. Table 2 displays the step-by-step outcomes of this recursive algorithm for finding t_{1A}*. As a result, $t_{1A}^* = 0.1213$ and by applying Eq. (34), we obtain $E[TCU(t_{1A}^*)]$ = \$13,334.92 are obtained. The behavior of $E[TCU(t_{1A})]$ in relation to t_{1A} is exhibited in Fig. 7. It shows that as t_{1A} deviates from t_{1A}* (i.e., 0.1213) in both directions, $E[TCU(t_{1A})]$ increases significantly.

Table 2
Step-by-step outcomes of the recursive algorithm for finding t_{1A}*

Iteration #	t_{1AU}	$e^{-\beta t_{1AU}}$	t_{1AL}	$e^{-\beta t_{1AL}}$	$E[TCU(t_{1AU})]$	$E[TCU(t_{1AL})]$
-	-	0	-	1	-	-
1	**0.2953**	0.7443	**0.0881**	0.9157	$14,241.46	$13,445.95
2	0.1563	0.8553	0.1132	0.8930	$13,403.96	$13,340.13
3	0.1293	0.8787	0.1194	0.8875	$13,339.27	$13,335.20
4	0.1232	0.8841	0.1209	0.8861	$13,335.17	$13,334.94
5	0.1218	0.8853	0.1212	0.8858	$13,334.94	$13,334.92
6	0.1214	0.8856	0.1213	0.8858	$13,334.92	$13,334.92
7	0.1214	0.8857	0.1213	0.8857	$13,334.92	$13,334.92
8	**0.1213**	0.8857	**0.1213**	0.8857	**$13,334.92**	**$13,334.92**

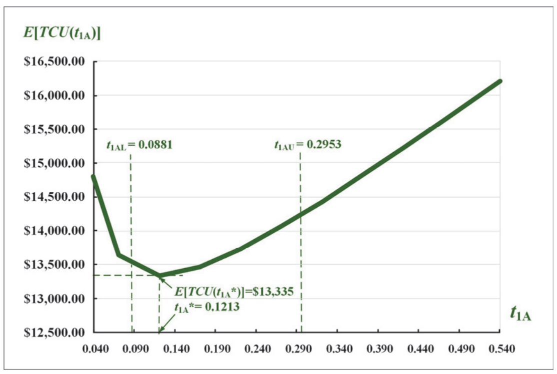

Fig. 7. The behavior of $E[TCU(t_{1A})]$ concerning t_{1A}

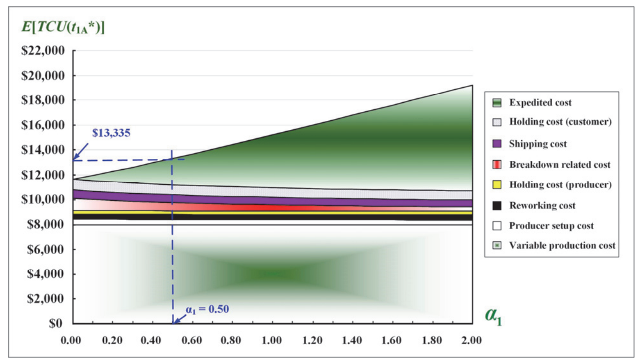

Fig. 8. The impact of changes in the accelerated ratio α_1 on each cost contributor of $E[TCU(t_{1A}*)]$

Fig. 8 illustrates the impact of changes in the accelerated ratio α_1 on each cost contributor of $E[TCU(t_{1A}*)]$. It indicates that the expedited cost increases drastically as α_1 rises, however, the stock holding cost and breakdown relevant cost decreases slightly; and it reconfirms that at $\alpha_1 = 0.5$, $E[TCU(t_{1A}*)] = \$13,335$. The behavior of t_{1A}* regarding α_1 is exhibited in Fig. 9. It indicates that t_{1A}* declines drastically, as α_1 increases; and it reconfirms our solution that at $\alpha_1 = 0.5$, $t_{1A}* = 0.1213$.

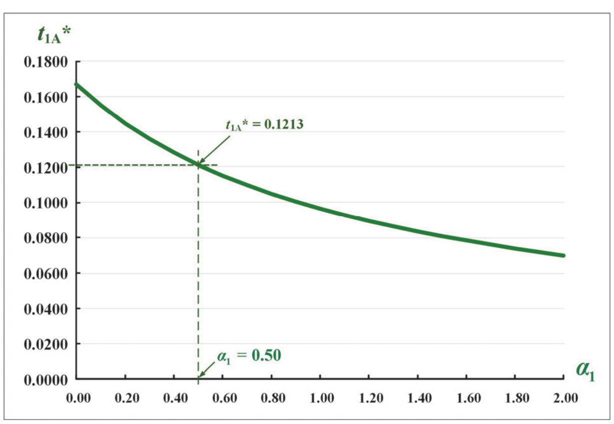

Fig. 9. The behavior of t_{1A}* regarding α_1

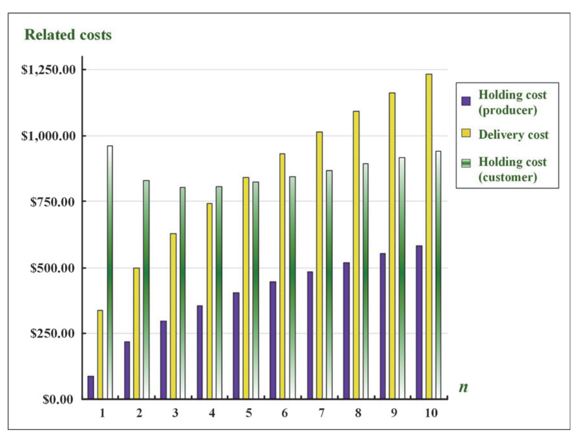

Fig. 10. The influence of differences in the frequency of delivery n on relevant system cost factors

The influence of differences in the frequency of delivery n on relevant system cost factors is depicted in Fig. 10. It shows that at $n = 1$, the customer's stock holding cost is highest among various n; both the producer's holding and delivery costs increase significantly as n goes up. Thus, we know that the frequency of delivery has more influence on the delivery cost than on in-house holding costs. Fig. 11 depicts the effect of changes in mean-time-to-breakdown $1/\beta$ on $E[TCU(t_{1A}^*)]$. It reveals that $E[TCU(t_{1A}^*)]$ declines, as $1/\beta$ rises; especially when $1/\beta \geq 0.25$ (i.e. annual breakdown instances $\beta \leq 4$), $E[TCU(t_{1A}^*)]$ starts to radically drop; and finally it reaches \$12,714, as $1/\beta$ approaches infinite (i.e., almost no breakdown occurrence in the fabrication process). Fig. 11 also reconfirms the solution of our example: $E[TCU(t_{1A}^*)]$ = \$13,335 at $1/\beta = 1$. Fig. 12 illustrates the detailed contributors to the system cost $E[TCU(t_{1A}^*)]$. It shows that a 16.01% is contributed by the expedited fabrication cost, rate, 5.01%, and 3.00% are from breakdown and rework relevant costs, and 4.73% and 6.02% of $E[TCU(t_{1A}^*)]$ are regarding the product distribution and customer's stock holding costs, etc.

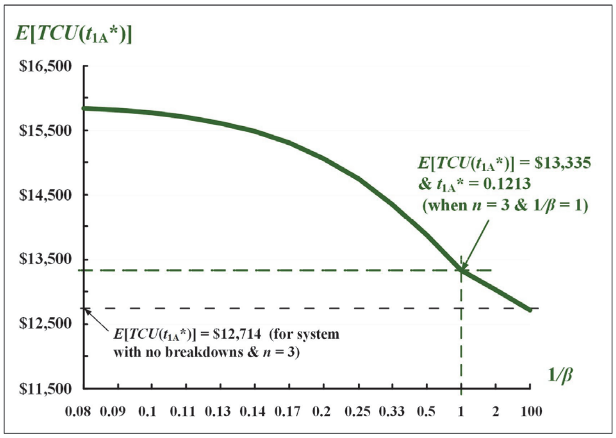

Fig. 11. The effect of changes in $1/\beta$ on $E[TCU(t_{1A}^*)]$

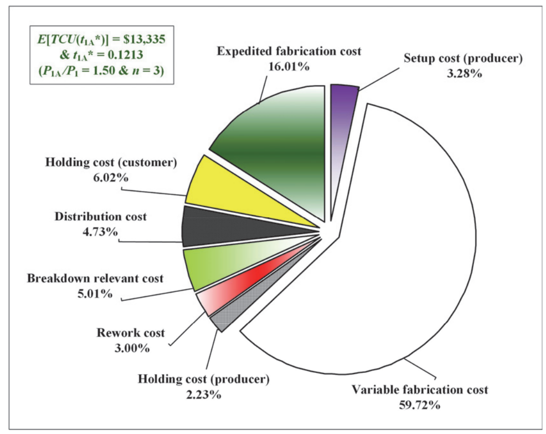

Fig. 12. The detailed contributors of $E[TCU(t_{1A}^*)]$

The joint impact of variations in the accelerated rate α_1 and random defective rate x on the rework cost is demonstrated in Fig. 13. It indicates that as x increases, the rework cost boosts radically; and as α_1 rises, the rework cost goes up slightly. Hence, we know that x has more impact on the rework cost than that of α_1. Fig. 14 depicts the combined influence of changes in the accelerated rework/fabrication related ratios C_{RA}/C_A and α_1 on $E[TCU(t_{1A}^*)]$. It exposes that as the ratio of C_{RA}/C_A increases, $E[TCU(t_{1A}^*)]$ goes up; and as the accelerated rate α_1 increases, $E[TCU(t_{1A}^*)]$ boosts severely. Thus, α_1 has more impact on $E[TCU(t_{1A}^*)]$ than that of C_{RA}/C_A. The effect of differences in the frequency of delivery n on $E[TCU(t_{1A}^*)]$ is exhibited in Fig. 15. It specifies that starting from $n > 2$ as n increases, $E[TCU(t_{1A}^*)]$ boosts considerably. Fig. 16 illustrates the combined impact of changes in the accelerated rate α_1 and mean- time-to-breakdown $1/\beta$ on optimal uptime t_{1A}^*. It exposes that t_{1A}^* declines significantly, as both $1/\beta$ and α_1 increases.

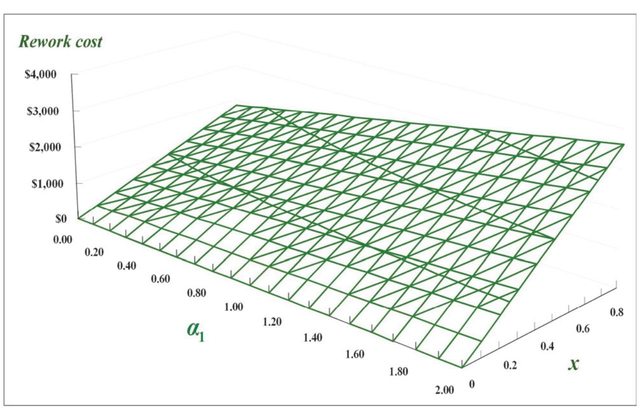

Fig. 13. The joint impact of variations in α_1 and x on the rework cost

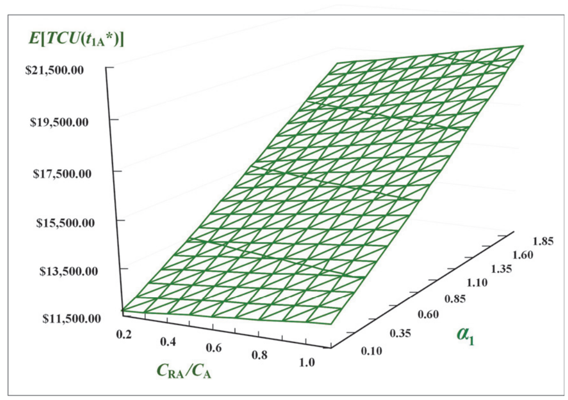

Fig. 14. The combined influence of changes in C_{RA}/C_A and α_1 on $E[TCU(t_{1A}^*)]$

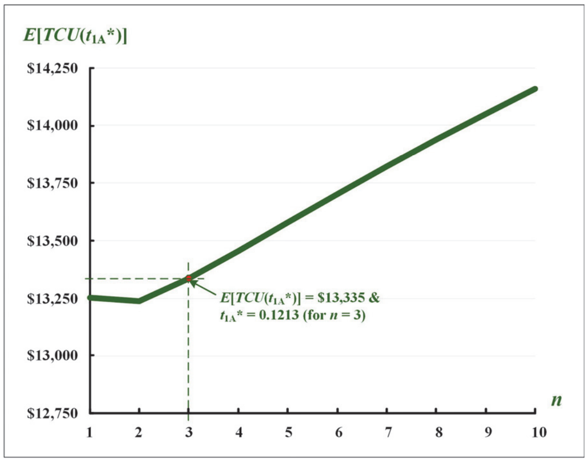

Fig. 15. The effect of differences in the frequency of delivery n on $E[TCU(t_{1A}^*)]$

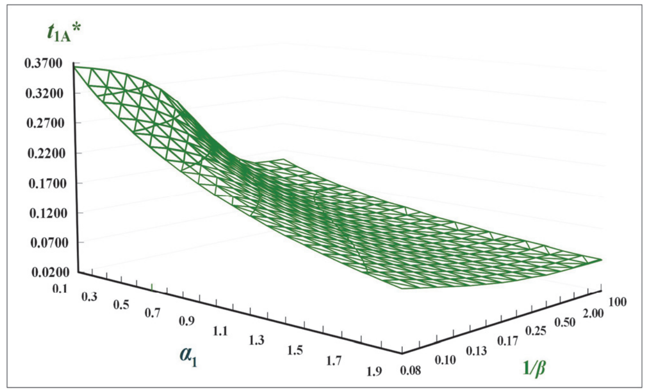

Fig. 16. The combined impact of changes in $1/\beta$ and α_1 on t_{1A}^*

5. Conclusions

This study explores the optimal fabrication runtime for a buyer-vendor incorporated system featuring repairable items, stochastic breakdown, accelerated rate, and multi-delivery strategy. We build a model to characterize the aforementioned features in the system, utilized formulation and analyses to derive the function of total system cost, and employed the optimization method and a recursive algorithm to find the optimal (i.e., cost minimization) fabrication runtime for the system. An example numerically illustrates how our model, method, and algorithm work (refer to the numerical demonstration section and Table 2). It also reveals the capability of our model in analyzing the impact of each and/or joint feature(s) (e.g., the breakdown, accelerated rate, rework, multi-delivery strategy) on the system's utilization, optimal runtime, total expenses, and individual cost contributor (refer to Figs. 7 to 16) to assist in managerial decision making, and hence, enabling the production firms to gain competitive advantage.

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Appendix – A

Details on obtaining Eq. (34) are as follows.

Substitute equations (19), (29), and (31) in Eq. (30), together with additional derivation efforts, we gain the following:

$$E\left[TCU\left(t_{1A}\right)\right]=\left[\frac{\lambda}{1+\dfrac{\lambda g\left(1-e^{-\beta t_{1A}}\right)}{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}}\right]\cdot\left[\begin{array}{l}\dfrac{\left[\left(1+\alpha_2\right)K\right]}{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}+\dfrac{nK_1}{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}+\left(1+\alpha_3\right)C+\left(1+\alpha_3\right)C_R E[x]\\ +C_T+h_3 g\left(e^{-\beta t_{1A}}\right)+\dfrac{E[x]^2\left(t_{1A}\right)\left(1+\alpha_1\right)P_1\left(h_1-h\right)}{2\left[\left(1+\alpha_1\right)P_2\right]}+\dfrac{C_T\lambda g\left(1-e^{-\beta t_{1A}}\right)}{t_{1A}\left[\left(1+\alpha_1\right)P_1\right]}\\ +\dfrac{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]\left(h_2-h\right)\left(1-y_1\right)}{2n\lambda}+\dfrac{g}{2n}\left(h_2-h\right)\left(1-y_1\right)\left(1-e^{-\beta t_{1A}}\right)\\ +\dfrac{h\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}{2\lambda}\left[1+\dfrac{\lambda E[x]}{\left(1+\alpha_1\right)P_2}\right]+\dfrac{hg}{t_{1A}}\left[-t_{1A}e^{-\beta t_{1A}}-\dfrac{1}{\beta}e^{-\beta t_{1A}}+\dfrac{1}{\beta}\right]\\ +\dfrac{M\left(1-e^{-\beta t_{1A}}\right)}{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}+\dfrac{C_1\lambda g\left(1-e^{-\beta t_{1A}}\right)}{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}+\dfrac{g}{2}h\left(1-y_1\right)\left(1-e^{-\beta t_{1A}}\right)\\ +\dfrac{h_2\lambda g^2\left(1-e^{-\beta t_{1A}}\right)}{2\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}+h_3 g\left[y_1+\dfrac{\lambda g}{\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}\right]\left(1-e^{-\beta t_{1A}}\right)\\ +\dfrac{h_2\left(t_{1A}\right)\left[\left(1+\alpha_1\right)P_1\right]}{2}\left(\dfrac{y_1}{\lambda}\right)+\dfrac{g}{2}h_2\left(1+y_1\right)\left(1-e^{-\beta t_{1A}}\right)\end{array}\right] \tag{A-1}$$

Let v_0, v_1, v_2, v_3, v_4, v_5, and v_6 denote the following:

$$v_0=\frac{(1+\alpha_2)K}{(1+\alpha_1)P_1}+\frac{nK_1}{(1+\alpha_1)P_1}$$

$$v_1=\frac{M}{(1+\alpha_1)P_1}+\frac{C_T\lambda g}{(1+\alpha_1)P_1}+\frac{C_1\lambda g}{(1+\alpha_1)P_1}+\frac{h_3\lambda g^2}{(1+\alpha_1)P_1}+\frac{h_2\lambda g^2}{2(1+\alpha_1)P_1}+\frac{hg}{\beta}$$

$$v_2=-hg$$

$$v_3=-\frac{M}{(1+\alpha_1)P_1}-\frac{C_T\lambda g}{(1+\alpha_1)P_1}-\frac{C_1\lambda g}{(1+\alpha_1)P_1}-\frac{h_3\lambda g^2}{(1+\alpha_1)P_1}-\frac{h_2\lambda g^2}{2(1+\alpha_1)P_1}-\frac{hg}{\beta}$$

$$v_4=\frac{g}{2}\left[h(1-y_1)+\frac{(h_2-h)(1-y_1)}{n}+(h_2+2h_3)(1+y_1)\right]$$

$$z_5=\frac{E[x]^2\left[(1+\alpha_1)P_1\right](h_1-h)}{2\left[(1+\alpha_1)P_2\right]}+\frac{\left[(1+\alpha_1)P_1\right](h_2-h)(1-y_1)}{2n\lambda}$$
$$+\frac{h_2\left[(1+\alpha_1)P_1\right]}{2}\left(\frac{y_1}{\lambda}\right)+\frac{h\left[(1+\alpha_1)P_1\right]}{2\lambda}\left[1+\frac{\lambda E[x]}{(1+\alpha_1)P_2}\right]$$

$$v_6=\left[(1+\alpha_3)C\right]+\left[(1+\alpha_3)C_R\right]E[x]+C_T$$

and

$$y_1=\frac{\lambda}{(1+\alpha_1)}\left[\frac{1}{P_1}+\frac{E[x]}{P_2}\right].$$

Then, Eq. (A-1) can be rearranged as Eq. (34) as follows:

$$E\left[TCU(t_{1A})\right]=\left[\frac{\lambda}{1+\frac{\lambda g\left(1-e^{-\beta t_{1A}}\right)}{(t_{1A})\left[(1+\alpha_1)P_1\right]}}\right]\left[\frac{v_0}{t_{1A}}+\frac{v_1}{t_{1A}}+v_2e^{-\beta t_{1A}}+\frac{v_3e^{-\beta t_{1A}}}{t_{1A}}-v_4e^{-\beta t_{1A}}+v_4+v_5t_{1A}+v_6\right] \quad (34)$$

Appendix – B

Apply the first- and second-derivatives of $E[TCU(t_{1A})]$, we gain the following:

$$\frac{dE\left[TCU(t_{1A})\right]}{d(t_{1A})}=\frac{\lambda(1+\alpha_1)P_1}{\left[(1+\alpha_1)P_1(t_{1A})+\lambda g\left(1-e^{-\beta t_{1A}}\right)\right]^2}\cdot$$
$$\begin{bmatrix}-(v_0+v_1)\left[(1+\alpha_1)P_1+e^{-\beta t_{1A}}\lambda g\beta\right]-(v_4+v_6)(\lambda g)\left(e^{-\beta t_{1A}}\beta(t_{1A})+e^{-\beta t_{1A}}-1\right)\\ +(v_2-v_4)\left[-e^{-\beta t_{1A}}\beta(t_{1A})^2(1+\alpha_1)P_1-e^{-\beta t_{1A}}\beta(t_{1A})\lambda g-e^{-2\beta t_{1A}}\lambda g+e^{-\beta t_{1A}}\lambda g\right]\\ +v_3\left[-e^{-\beta t_{1A}}\beta(t_{1A})(1+\alpha_1)P_1-e^{-\beta t_{1A}}\beta\lambda g-e^{-\beta t_{1A}}(1+\alpha_1)P_1\right]\\ +v_5(t_{1A})\left[(t_{1A})(1+\alpha_1)P_1+2\lambda g\left(1-e^{-\beta t_{1A}}\right)-e^{-\beta t_{1A}}(t_{1A})\lambda g\beta\right]\end{bmatrix} \quad \text{(B-1)}$$

and

$$\frac{d^2E\left[TCU\left(t_{1\mathrm{A}}\right)\right]}{d\left(t_{1\mathrm{A}}\right)^2}=\frac{\lambda\left[\left(1+\alpha_1\right)P_1\right]}{\left[\left(t_{1\mathrm{A}}\right)\left[\left(1+\alpha_1\right)P_1\right]+\lambda g\left(1-e^{-\beta t_{1A}}\right)\right]^3}\cdot$$
$$\left[\begin{aligned}&\left(v_0+v_1\right)\left[\begin{aligned}&\left(\lambda g\beta\right)^2\left(e^{-2\beta t_{1A}}+e^{-\beta t_{1A}}\right)+e^{-\beta t_{1A}}\left(t_{1\mathrm{A}}\right)\left(\lambda g\right)\beta^2\left[\left(1+\alpha_1\right)P_1\right]\\&+4e^{-\beta t_{1A}}\left(\lambda g\right)\beta\left[\left(1+\alpha_1\right)P_1\right]+2\left[\left(1+\alpha_1\right)P_1\right]^2\end{aligned}\right]\\&+\left(v_2-v_4\right)e^{-2\beta t_{1A}}\left[\begin{aligned}&\left(\lambda g\beta\right)^2t_{1\mathrm{A}}\left(1+e^{\beta t_{1A}}\right)+2\beta\left(\lambda g\right)^2\left(1-e^{\beta t_{1A}}\right)+e^{\beta t_{1A}}\beta^2\left(t_{1\mathrm{A}}\right)^3\left[\left(1+\alpha_1\right)P_1\right]^2\\&+\lambda g\left(1+\alpha_1\right)P_1\left[\beta^2\left(t_{1\mathrm{A}}\right)^2\left(2e^{\beta t_{1A}}+1\right)+2\beta\left(t_{1\mathrm{A}}\right)\left(2-e^{\beta t_{1A}}\right)+2\left(1-e^{\beta t_{1A}}\right)\right]\end{aligned}\right]\\&+\left(v_4+v_6\right)\left(\lambda g\right)\left[\begin{aligned}&\left(\lambda g\right)\beta^2\left(t_{1\mathrm{A}}\right)\left(e^{-2\beta t_{1A}}+e^{-\beta t_{1A}}\right)+2\left(\lambda g\beta\right)\left(e^{-2\beta t_{1A}}-e^{-\beta t_{1A}}\right)\\&+e^{-\beta t_{1A}}\left[\left(1+\alpha_1\right)P_1\right]\left[\beta^2\left(t_{1\mathrm{A}}\right)^2+2\beta\left(t_{1\mathrm{A}}\right)+2\right]-2\left[\left(1+\alpha_1\right)P_1\right]\end{aligned}\right]\\&+v_3\left(e^{-\beta t_{1A}}\right)\left[\begin{aligned}&\left(\lambda g\beta\right)^2\left(e^{-\beta t_{1A}}+1\right)+\beta^2\left(t_{1\mathrm{A}}\right)\left(\lambda g\right)\left[\left(1+\alpha_1\right)P_1\right]\left(2+e^{-\beta t_{1A}}\right)\\&+\left[\left(1+\alpha_1\right)P_1\right]^2\left[\beta^2\left(t_{1\mathrm{A}}\right)^2+2\beta\left(t_{1\mathrm{A}}\right)+2\right]+2\beta\left(\lambda g\right)\left[\left(1+\alpha_1\right)P_1\right]\left(1+e^{-\beta t_{1A}}\right)\end{aligned}\right]\\&+v_5\left(\lambda g\right)\left[\begin{aligned}&2e^{-2\beta t_{1A}}\left(\lambda g\right)+e^{-2\beta t_{1A}}\left(t_{1\mathrm{A}}\right)^2\left(\lambda g\right)\beta^2+4\left(t_{1\mathrm{A}}\right)\left(\lambda g\right)\beta\left(e^{-2\beta t_{1A}}-e^{-\beta t_{1A}}\right)\\&+e^{-\beta t_{1A}}\left(t_{1\mathrm{A}}\right)^2\left(\lambda g\right)\beta^2+e^{-\beta t_{1A}}\left(t_{1\mathrm{A}}\right)^3\beta^2\left[\left(1+\alpha_1\right)P_1\right]-4e^{-\beta t_{1A}}\left(\lambda g\right)+2\left(\lambda g\right)\end{aligned}\right]\end{aligned}\right] \tag{B-2}$$

Because the first term on the right-hand side (RHS) of Eq. (B-2) is positive, so $E[TCU(t_{1\mathrm{A}})]$ is convex if one can prove Eq. (B-3) holds.

$$\delta\left(t_{1\mathrm{A}}\right)=\frac{\begin{aligned}&-\left(v_0+v_1\right)\left[\left(\lambda g\beta\right)^2\left(e^{-2\beta t_{1A}}+e^{-\beta t_{1A}}\right)+\left(1+\alpha_1\right)P_1\left[4e^{-\beta t_{1A}}\left(\lambda g\beta\right)+2\left(1+\alpha_1\right)P_1\right]\right]\\&-\left(v_2-v_4\right)\left(e^{-2\beta t_{1A}}\right)\left(1-e^{\beta t_{1A}}\right)\left[2\beta\left(\lambda g\right)^2+2\left(\lambda g\right)\left(1+\alpha_1\right)P_1\right]\\&-\left(v_4+v_6\right)\left(\lambda g\right)\left[2\left(\lambda g\right)\beta\left(e^{-2\beta t_{1A}}-e^{-\beta t_{1A}}\right)-2\left(1+\alpha_1\right)P_1\left(1-e^{-\beta t_{1A}}\right)\right]\\&-v_3\left(e^{-\beta t_{1A}}\right)\left[\left(\lambda g\beta\right)^2\left(1+e^{-\beta t_{1A}}\right)+2\left(1+\alpha_1\right)P_1\beta\lambda g\left(1+e^{-\beta t_{1A}}\right)+2\left[\left(1+\alpha_1\right)P_1\right]^2\right]\\&-v_5\left(\lambda g\right)^2\left(2e^{-2\beta t_{1A}}-4e^{-\beta t_{1A}}+2\right)\end{aligned}}{\begin{aligned}&\left[\left(v_0+v_1\right)\right]\left[e^{-\beta t_{1A}}\left(\lambda g\right)\beta^2\left[\left(1+\alpha_1\right)P_1\right]\right]\\&+\left(v_2-v_4\right)\left(e^{-2\beta t_{1A}}\right)\left[\begin{aligned}&\left(\lambda g\beta\right)^2\left(1+e^{\beta t_{1A}}\right)+e^{\beta t_{1A}}\beta^2t_{1\mathrm{A}}^2\left[\left(1+\alpha_1\right)P_1\right]^2\\&+\left(\lambda g\right)\left(1+\alpha_1\right)P_1\left[2e^{\beta t_{1A}}\beta^2t_{1\mathrm{A}}+\beta^2t_{1\mathrm{A}}+4\beta-2e^{\beta t_{1A}}\beta\right]\end{aligned}\right]\\&+\left(v_4+v_6\right)\left(\lambda g\beta\right)\left[\left(\lambda g\beta\right)\left[e^{-2\beta t_{1A}}+e^{-\beta t_{1A}}\right]+y_0\left(1+\alpha_1\right)P_1\left(e^{-\beta t_{1A}}\beta t_{1\mathrm{A}}+2e^{-\beta t_{1A}}\right)\right]\\&+v_3\left(e^{-\beta t_{1A}}\right)\left(1+\alpha_1\right)P_1\left[2\beta^2\lambda g+e^{-\beta t_{1A}}\beta^2\lambda g+\beta^2t_{1\mathrm{A}}\left(1+\alpha_1\right)P_1+2\beta\left(1+\alpha_1\right)P_1\right]\\&+v_5\lambda g\left[\lambda g\beta\left(e^{-2\beta t_{1A}}t_{1\mathrm{A}}\beta+4e^{-2\beta t_{1A}}+e^{-\beta t_{1A}}t_{1\mathrm{A}}\beta-4e^{-\beta t_{1A}}\right)+e^{-\beta t_{1A}}t_{1\mathrm{A}}^2\beta^2\left(1+\alpha_1\right)P_1\right]\end{aligned}}>t_{1\mathrm{A}}>0$$

(B-3)

Appendix – C

Table C
More convexity tests of $E[TCU(t_{1\mathrm{A}})]$ with a wider choice of β values

β	$\delta(t_{1\mathrm{AL}})$	$t_{1\mathrm{AL}}$	$\delta(t_{1\mathrm{AU}})$	$t_{1\mathrm{AU}}$
11	0.0396	0.0185	0.9517	0.2904
8	0.0531	0.0246	0.6303	0.2906
5	0.0811	0.0366	0.4678	0.2910
4	0.0985	0.0435	0.4402	0.2913
3	0.1253	0.0531	0.4276	0.2917
2	0.1727	0.0672	0.4389	0.2926
1	**0.2886**	**0.0881**	**0.5205**	**0.2953**
0.5	0.4788	0.1020	0.6965	0.3006
0.01	3.8167	0.1183	4.3397	0.6320

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