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Production control problem for multi-product multi-resource make-to-stock systems

Sinem Özkan[a*], Önder Bulut[b] and Mehmet Cemali Dinçer[b]

[a]*Department of Industrial Engineering, İzmir Democracy University, İzmir, Turkey*
[b]*Department of Industrial Engineering, Yaşar University, İzmir, Turkey*

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A B S T R A C T

Most of today's production systems are working with parallel production resources to increase throughput rate due to the increase in high variability in demand and product mix. Effective control and performance evaluation of such systems is of paramount importance to minimize production and inventory-related costs. We examine a production-inventory system featuring parallel production resources capable of producing various products. In many industries such as automotive, white goods, electronics, and paint, multiple/parallel production resources are widely used to produce the ideal amount and satisfy incoming demands for distinct products. In this study, shortage cost is not restricted to only one type and both lost sales and backordering cases are analyzed. In order to analyze the optimal production policies' behavior, we initially formulate dynamic programming models for both lost sales and backordering systems, treating them as Markov Decision Processes. Subsequently, we solve these models using the value iteration algorithm. Given the challenges posed by the curse of dimensionality in the value iteration algorithm, we suggest alternative heuristic production policies. These policies extend the existing ones described for multi-item single-resource make-to-stock (MTS) systems to accommodate multiple resources. We construct simulation models to assess the efficacy of the heuristic policies, conducting comparisons of their performance against both the optimal policy and among one another. To the best of our knowledge, there has been no exploration of scenarios involving multiple production resources concurrently manufacturing distinct products in a MTS environment. Hence, this study serves as an extension to the examination of multi-item, multi-production resource MTS systems.

1. Introduction

Most production systems face randomness in different parts of the process. Two of the most common random effects are due to production and inter-demand times. Production planners/controllers in make-to-stock (MTS) systems aim to achieve a balance between holding, shortage, and production-related costs. With the growing variety of products to be produced, the significance of minimizing costs and implementing efficient control over production resources becomes increasingly apparent. Due to the increase in product variety and their demand rates, most of today`s production systems operate with parallel production resources to reduce production lead-time and hence response time. Effective control and performance evaluation of the above-mentioned multi-resource stochastic production-inventory systems are of great importance for minimizing production and inventory related costs.

This paper focuses on MTS systems with a single operation, incorporating random inter-demand and production times. We study both lost sales and backordering cases as well as multi-resource (multi-server) and multi-item. With this broad perspective, we believe that a variety of real-life MTS systems are considered. To better understand the problem environment from the practitioners' point of view, we take an example from the electronics industry. Some television manufacturers first assemble the raw materials and WIP items to hold them in a common buffer stock. All products in this buffer are identical

* Corresponding author
E-mail sinem.ozkan@idu.edu.tr (S. Özkan)
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and semi-finished. Upon demand from customers (end-users), products are withdrawn from the buffer and differentiated according to customer needs, such as screen quality, color preferences, etc. This strategy is known as Delayed Differentiation in the literature (Lee & Tang, 1997). The demand for products in the buffer stock (pre-differentiation) is the cumulative demand of all the customers. If production up to the common stock needs to be controlled and the process is modeled as a single operation with random operation times (due to human factors, breakdowns, etc.), this model would be a single item stochastic MTS model. The demand process for the common stock can be represented as a Poisson process, given that it is the aggregate of all customer demands. Also, some TV manufacturers might have multiple, parallel resources to feed the common stock. Such cases can be modelled as multi-server MTS systems. Lastly, the common stock of different customer groups or brands demanding TVs in different technologies, sizes, etc., might be different. Such cases are examples for multi-item MTS systems. Moreover, a real-life example from the paint industry for a similar production-inventory system is explained in detail in Özkan and Bulut (2022).

This article examines a production-inventory system featuring parallel production resources capable of manufacturing various products. These resources are non-preemptive and have the capability to manufacture any type of product, requiring only one production resource to complete the production of a product. Simultaneously, multiple products can be produced in parallel. The production times for each product on a production line follow independent and identically distributed exponential random variables. Each product undergoes a demand process with exponentially distributed interarrival times. Furthermore, shortage cost is not confined to a single type, and both lost sales and backordering cases are analyzed. In this study, we initially investigate the optimal production policy's behavior for both backordering and lost sales cases. Subsequently, alternative policies are proposed. Finally, we compare their performances with the optimal policy and among themselves.

In multi-item production settings, production resources adhere to a production policy, which comprises two primary decisions: deciding when and how much to produce, as well as choosing the specific products for production. Existing literature on this topic delineates production policies into two distinct control strategies: an idleness policy that governs the on/off status of production, and a scheduling policy that determines the order in which products are produced (Tiemessen et al., 2017). In existing literature, the base stock policy (BSP) is considered optimal for single-item single-production resource MTS systems that do not involve start-up costs (Ha 1997a; Ha 1997b), they employ the BSP to determine when to start or stop production in multi-item single-production resource MTS systems. They differ only in the priority rule that determines production priority. In this study, as *an idleness policy*, we use a pre-proposed well-performing policy, *the extended-two-critical-number policy* (ETCNP), for a single-item multi-production resource system (Özkan and Bulut, 2020). ETCNP has two control parameters: CL_1 is the production trigger point and it also bounds the number of production lines that can be activated for multi-line systems. CL_2 is the produce-up-to level (the maximum inventory level) such that $CL_2 > CL_1$. However, when the start-up cost is negligible, the optimal difference between production control levels is usually one, otherwise more than one. Therefore, a BSP is a near-optimal policy for single-item multi-production resource systems, and we again use the BSP to determine when to start or stop production even in a multi-production resource case.

The used methodology and contribution of this study are listed as:

- In the related literature, there are extensive works on the problem consisting of a single production resource where multiple products are produced (e.g., Zheng and Zipkin, 1990; Veatch and Wein, 1996; Ha, 1997c; Sanajian et al., 2010; Tiemessen, 2017). This study is the inaugural exploration to simultaneously address the presence of multiple non-identical products and parallel production resources. This assumption aligns with real-life scenarios, as production facilities frequently have the capacity to concurrently manufacture multiple products.
- Most of the studies on the control of multi-item MTS systems are for backordering environment (e.g., Zheng and Zipkin, 1990; Perez and Zipkin, 1997; Ha, 1997c; Sanajian et al., 2010; Tiemessen, 2017). We relax this assumption and consider both lost sales and backordering cases for multi-item multi-production resource MTS systems.
- For both scenarios involving lost sales and backordering, we formulate the dynamic programming model for the system as a Markov Decision Process. The models are subsequently solved using the value iteration algorithm. In light of the curse of dimensionality, we analyze the optimal production policy's structure using a numerical example that centers around a smaller problem instance.
- Since alternative approaches become prominent for large problem instances, we propose alternative heuristic production policies for both lost sales and backordering cases benefiting from pre-proposed policies in the literature. We compare them on a wide range of problem instances. We show that most of our proposed production policies perform well with a broad numerical experiment.

The rest of this paper is structured as follows: Section 2 provides a thorough literature review. Sections 3.1 and 3.2 introduce $M/M/s$ multi-item MTS queues for backordering and lost sales environments, respectively. The problem formulations are provided, and the optimal policy structure is discussed with numerical examples. Section 4 explains our extended scheduling policies applicable to multi-item multi-production line systems for both backordering and lost sales cases. In Section 5, experimental studies are conducted to assess the performance of the new heuristic policies. Finally, Section 6 concludes with remarks and suggestions for future research.

2. Literature Review

The literature extensively explores the single-product version of make-to-stock (MTS) queue problems. Earlier studies predominantly focused on systems with a single production resource and a single demand class, incorporating start-up and/or shut-down costs. Most of these earlier studies pertain to backordering environments, as evidenced by works such as Gavish and Graves (1981), Lee and Srinivasan (1987), Altıok (1989), and Ha (1997b). Conversely, studies involving lost sales assume deterministic, Markovian, or generally distributed processing times, with examples including De Kok (1985), De Kok and Tijms (1985), Baek and Moon (2016), Jose and Nair (2017), Yue and Qin (2019), and Özkan and Bulut (2022). The analyses in these studies often rely on queueing and inventory theory techniques. In more recent investigations, the production-inventory problem is viewed as a control problem, leading to the development of Markov Decision Process (MDP) models. This shift is evident in studies such as Bulut and Fadıloğlu (2011) and Özkan and Bulut (2020).

The literature has explored the multi-product version of MTS queue problems, with attention directed towards two distinct issues. The first involves the optimal and/or near-optimal production control problem, while the second revolves around the stochastic economic lot scheduling problem (SELSP). Investigations addressing the first problem seek to delineate the optimal policy structure. However, confronted with the curse of dimensionality, these studies typically examine optimal policies for small-scale instances. As an alternative for larger instances, they introduce heuristic production policies and assess their performance in comparison to the optimal approach and pre-existing policies.

Within the relevant literature, production policies typically consist of two distinct control policies: when production is active or idle is governed by an idleness policy, and the next product to be produced is selected by a scheduling policy. Prior research in this domain commonly utilizes base stock policies (BSPs) to decide when to initiate or cease production, differing only in the priority rule dictating production precedence. Consequently, studies in literature introduce novel approaches to scheduling priority and conduct comparative analyses. It is worth noting that the BSP is considered optimal for single-item single-production resource MTS systems without startup costs.

Zheng and Zipkin (1990) conducted one of the initial studies in literature addressing the control of multi-item MTS systems. They study a production system consisting of two identical products, a common production resource, and backorders. Demands for the products form independent Poisson processes. The production resource produces only one unit of either product at a time. Production times are assumed to be independent and identically distributed exponential random variables. Control of the entire system is achieved by deciding when and which product to produce. They employ a BSP, in short $(CL_2 - 1, CL_2)$ policy, to determine when to start or stop production. To decide which product is produced, they consider a dynamic longest queue policy. For a specific product, when the inventory position falls below the maximum inventory level CL_2, an order is placed and sent to the production resource. As demand arrives, the outstanding orders form a queue at the production. The production resource remains on until all orders are satisfied, i.e., until there are CL_2 units of inventory of both products. If there is at least one product in the queue, production occurs. However, if there is more than one type of product, the production resource chooses which product is produced based on the longest queue, that is, the product is chosen to have the lowest inventory level. In their 1990 study, Zheng and Zipkin compare the performance of the longest queue policy to the static FCFS for two identical products when the production is controlled with a base stock policy. They demonstrate that the dynamic longest queue policy surpasses the static FCFS. Subsequently, in 1995, Zipkin reevaluate the performance of the dynamic longest queue policy for systems with arbitrarily distributed production times and more than two products.

Wein (1992) considers a $G/G/1$ queue with non-identical products and develops a scheduling policy by using the heavy-traffic approximation. He proposes a well-performing production policy composed of an aggregate BSP and a dynamic scheduling policy which is called the $b\mu/h\mu$ rule in the literature. According to the $b\mu/h\mu$ rule, when at least two products have backorders, the product which has the biggest value of the index $b_i\mu_i$ is chosen (given priority) to produce. Here, for any product i, b_i is the backordering cost rate and μ_i is the production rate. When all the products have no backorder, but their inventory levels are less than their base-stock (produce-up-to) levels, the product which has the smallest value of the index $h_i\mu_i$ is chosen to produce. h_i is the holding cost rate for product i. Veatch and Wein (1996) work on many different couples of scheduling and idleness policies. They demonstrate that the production policy composed of the aggregate BSP of Wein (1992) and the dynamic service time look-ahead policy proposed by Perez and Zipkin (1997) perform well. They also obtain the optimal policy for the systems with two and three products in an $M/M/1$ queue.

Perez and Zipkin (1997) propose a dynamic scheduling policy structure named Myopic (P). It is related to the fully myopic policy of using the cost rate after one transition, which is the $b\mu/h\mu$ rule. The Myopic (P) rule considers the expected cost rate after one production time. According to Myopic (P), if a product is currently being produced, the next product is selected to be produced when the production of that product is completed. Production cannot be interrupted; hence this policy has a non-preemptive nature. Myopic (P) looks ahead one production time and calculates the cost difference between having one more unit of product i and not having it. Then, it allocates the production capacity to the product that has the smallest cost increment (if on-hand inventory increases) or the biggest cost decrement (if the number of backorders decreases). As an improvement, the use of sojourn time as a look-ahead time has been suggested in the Myopic (T) policy. They demonstrate that the Myopic (T) policy outperforms the $b\mu/h\mu$ rule and the dynamic service time look-ahead policy.

Ha (1997c) suggests a novel dynamic scheduling policy named the switching rule. The switching rule always awards higher priority to backordered products. Among the products that are backordered, priority is given to the one with the largest $b_i\mu_i$. Let denote the base stock level of product i by CL_i. If no product is backordered, priority is given to the one with the largest $b_i\mu_i(1-x_i/CL_i)$. The quantity $(1-x_i/CL_i)$ can be interpreted as the proportion of unfilled base stock and the larger it is, the closer a product will be to being backordered. Numerical findings indicate that the switching rule exhibits a negligible optimality gap, and it performs better than the suggested two other scheduling rules. De Vericourt et al. (2000) generalizes the results of Ha (1997c) on the form of the optimal policy. They indicate that the Myopic (T) policy is optimal for states with backorders of the product with the lowest $b_i\mu_i$.

Sanajian et al. (2010) considers a problem consisting of a repair shop. In the preemptive case, it is shown with the experimental results that the Myopic (T) policy is generally well-performing. In a recent study, Tiemessen et al. (2017) proposes a new scheduling policy, the rolling horizon scheduling policy, and compares the proposed policy's performance with the scheduling policies suggested in the literature. They present that the rolling horizon scheduling policy outperforms the switching rule and the Myopic (T) scheduling policy.

There is also a vast literature that considers inventory lot sizing models such as the Economic Order Quantity (EOQ) and the Economic Production Quantity (EPQ). We direct the reader to the recent works of Afonso et al. (2023) and Chiu et al. (2023) and the related literature.

Table 1 summarizes the milestone works in the literature on multi-item production systems. In the control literature of multi-item MTS queues, the problem consisting of a single production resource where multiple products are produced in order has been extensively examined. To the best of our knowledge, no study considers multiple production resources where different products are produced in parallel. Therefore, this work provides an extension to the analysis of the multi-item multi-production resource MTS systems. The main contribution is to extend the control of multi-item MTS literature by considering multiple production resources for both pure lost sales and pure backordering environments. We obtain a value iteration solution of a Markov decision process. Insights into the optimal policy structure are revealed with the help of dynamic programming results. We also propose new heuristic production policies and compare their performances with the optimal one and among themselves for multi-item MTS systems.

The literature review section ends with a brief discussion of the Stochastic Economic Lot Scheduling Problem (SELSP). SELSP considers MTS systems with different products produced on a single production resource with production set-up times under non-deterministic demands or production times. Studies in the SELSP literature employ production policies to decide whether to continue production of the current product, switch to another product, or leave the production resource idle. The common purpose of these policy is to minimize expected total costs consisting of set-up, holding, and shortage costs. Most works in the related literature consider a fixed production sequence and a dynamic cycle length (see e.g., Bourland and Yano (1994), Gascon et al. (1994), Markowitz (2001), Smits et al. (2004), Grasman et al. (2008). Winands et al. (2011) conduct a comprehensive review of the literature on SELSP. The methods devised to address the SELSP have been formulated based on the assumption that setup times are significant.

Table 1
Related literature on multi-item make-to-stock (MTS) systems

Optimal Control of Multi-Item Production-Inventory Systems	
(Veatch & Wein, 1996; Ha, 1997; De Vericourt et al., 2000; Veatch & De Vericourt, 2003; Tiemessen et al., 2017)	$M/M/1$ MTS queue with backorders
(Veatch & Wein, 1996)	$M/M/1$ MTS queue with lost sales
Heuristic Production Policies for Multi-Item Production-Inventory Systems	
(Zheng & Zipkin, 1990; Veatch & Wein, 1996; Pena Perez & Zipkin, 1997; Ha, 1997; Tiemessen et al., 2017)	$M/M/1$ MTS queue with backorders
(Veatch & Wein, 1996)	$M/M/1$ MTS queue with lost sales
(Zipkin, 1995; Sanajian et al., 2010)	$M/G/1$ MTS queue with backorders
(Wein, 1992)	$G/G/1$ MTS queue with backorders

3. Dynamic Programming Formulations

3.1. Backordering Case

This section delves into a make-to-stock (MTS) production-inventory system employing multiple production resources to produce a variety of products. The production times for each product on a production line are assumed to follow independent and identically distributed exponential random variables. Additionally, the time between two consecutive demand arrivals for each product is assumed to follow an exponential distribution. In the event of a product's demand occurrence with no existing backorder, the arriving demand is promptly fulfilled. Conversely, if a backorder exists for the requested product, it is placed on backorder and fulfilled when the on-hand inventory level becomes positive. The system is formulated as a Markov Decision Process (MDP) using dynamic programming, and subsequently, the model is solved utilizing the value iteration algorithm, enabling an in-depth exploration of the optimal production policy through a numerical study. The underlying assumptions are outlined as follows:

- Production on a production resource cannot be stopped (non-preemption). All resources are capable of producing all products, and each production resource can handle only one item at a time.
- Production times are assumed to follow Exponential random variables.
- Multiple types of products are stored in their respective stocks, and each product undergoes a demand process with exponentially distributed interarrival times.
- All cost types are specific to each item. There exists a rate for inventory holding costs and a rate for backordering costs.
- Switching times and costs are negligible.
- There are plenty of raw materials. Only production capacity is limited.

Given these assumptions and the Markovian structure of both demand arrivals and production times, along with the presence of s parallel production resources, the production-inventory system is represented as an $M/M/s$ multi-item MTS queue. We formulate the corresponding production control problem as a continuous-time MDP. The parameters employed in the problem formulation are outlined in Table 2.

Table 2
Parameters used in dynamic programming formulation

s	Number of available production resources
μ_i	Production rate of item i
λ_i	Demand rate of item i
α	Discount rate
b_i	Backordering cost for item i
h_i	Holding cost for item i

Initially, the system state is characterized by two primary sets of variables designed to monitor events. Let $x_i(t)$, where $i = 1, \dots, |I|$, be the inventory level of item i at time t, and $y_i(t)$, where $i = 1, \dots, |I|$, denote the number of active resources producing item i at time t. In multi-item single-line systems, in addition to the inventory levels, the information of which product is currently being produced on the line is necessary to identify the inventory replenishment rate at time t. Here, however, we also ought to track the number of active lines (can be more than one) producing which item to determine the replenishment rate at time t. Due to Exponential production times and inter-demand arrival times assumptions, the structure of the problem is Markovian. Markov property refers to the memoryless property of a stochastic process. Hence, the system state definition is applied uniformly across the time dimension. Consequently, the state space is:

$$SS = \left\{ \left((x_1. \dots . x_{|I|}). (y_1. \dots . y_{|I|}) \right) | x_i \in \mathbb{Z}. i \in I. \sum_{i=1}^{|I|} y_i \leq s \right\} \tag{1}$$

To streamline the state representation, we introduce $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{x} = (x_1, \dots, x_{|I|})$ and $\boldsymbol{y} = (y_1, \dots, y_{|I|})$ as $|I|$-dimensional vectors. Through the Memoryless property, it is possible to only focus on an occurrence of either a production completion on a line or a demand arrival. As soon as any of these random events occur, the control indicates whether to keep the number of active lines at the same level or to activate more. The production decision is expressed as u_i, $i = 1, \dots, |I|$, representing the number of active resources on which item i is produced. As preemption is not allowed, the admissible decisions for each state $\boldsymbol{z} \in SS$ satisfy the following:

$$u_i \in \left\{ y_i. y_i + 1. \dots . \left(s - \sum_{i' \neq i} y_{i'} \right) \right\}. \qquad \forall i \in I. \sum_{i=1}^{|I|} y_i \leq s \tag{2}$$

Because production decision promptly alters the system state, to use a simple form, we define a new vector $\boldsymbol{u} = (u_1, \dots, u_{|I|})$ which is a $|I|$-dimensional vector. Hence, the new state will be $(\boldsymbol{x}, \boldsymbol{u}) = \left((x_1, \dots, x_{|I|}), (u_1, \dots, u_{|I|}) \right)$.

Under a control policy π, the original problem transforms into a production-inventory control problem in continuous time. Utilizing the uniformization technique introduced by Lippman (1975), we derive a discrete-time counterpart for this problem. The uniform transition rate can be defined as the maximum transition rate that the system can observe at any time and calculated as $v = s\max_i\{\mu_i\} + \sum_{i=1}^{|I|}\lambda_i$.

We now define the random events for the MDP formulation. The system's state can be altered by two distinct types of events. These are a production completion of an item in a production resource, and a demand arrival for an item. We furthermore define a new vector; $\boldsymbol{e_i}$, to simplify the transitions between the states. $\boldsymbol{e_i}$ is a unit vector of dimension $|I|$ with a value of 1 at position i.

Building upon the provided definitions, the optimal cost-to-go function for the equivalent discrete-time problem is expressed in equation (3). As it is stated by Bertsekas (2000), an optimal control policy π^* exists and can be obtained through the solution of the below cost-to-go function.

$$J(\boldsymbol{x}.\boldsymbol{y}) = \frac{1}{v+\alpha} \min_{\substack{y_i \le u_i \le (s-\sum_{i' \neq i} y_{i'}) \\ \forall i}} \Bigg\{ \sum_{i=1}^{|I|} h_i x_i^+ + \sum_{i=1}^{|I|} b_i x_i^- + \left(s\max_i\{\mu_i\} - \sum_{i=1}^{|I|} u_i\mu_i \right) J(\boldsymbol{x}.\boldsymbol{u}) + \sum_{i=1}^{|I|} T_{P_i}(\boldsymbol{x}.\boldsymbol{u}) + \sum_{i=1}^{|I|} T_{D_i}(\boldsymbol{x}.\boldsymbol{u}) \Bigg\} \tag{3}$$

where

$x_i^+ = max(0, x_i)$

$x_i^- = -min(0, x_i)$

$T_{P_i}(\boldsymbol{x},\boldsymbol{u}) = u_i\mu_i min\left\{J(\boldsymbol{x}+\boldsymbol{e_i},\boldsymbol{u}-\boldsymbol{e_i}), \min_{i' \in I}\{\boldsymbol{x}+\boldsymbol{e_i},\boldsymbol{u}-\boldsymbol{e_i}+\boldsymbol{e_{i'}}\}\right\}$

$T_{D_i}(\boldsymbol{x},\boldsymbol{u}) = \lambda_i J(x-\boldsymbol{e_i},\boldsymbol{u})$

Eq. (3) minimizes the expected long-run discounted system cost by deciding how many production resources should produce which item at any given state. In (3), apart from the holding cost, a backordering cost is incurred for backlogged items. The next term is $\left(s\max_i\{\mu_i\} - \sum_{i=1}^{|I|} u_i\mu_i\right) J(\boldsymbol{x},\boldsymbol{u})$ represents fictitious self-transitions introduced by uniformization. In this self-transition term, the state is denoted as $(\boldsymbol{x},\boldsymbol{u})$ instead of $(\boldsymbol{x},\boldsymbol{y})$ because the production decision $\boldsymbol{u}$ instantaneously modifies $\boldsymbol{y}$.

The following operators are $T_{P_i}(\boldsymbol{x},\boldsymbol{u})$ and $T_{D_i}(\boldsymbol{x},\boldsymbol{u})$ are for production and demand-related transitions, respectively. $T_{P_i}(\boldsymbol{x},\boldsymbol{u})$ represents the transitions initiated by the completion of production for item i. The minimization operation is defined with a rate $u_i\mu_i$, signifying the decision made at the time of production completion for item i: first, the inventory is replenished, and then a determination is made regarding the item to be produced on the resource that has just finished processing. The operator $T_{D_i}(\boldsymbol{x},\boldsymbol{u})$ corresponds to the transitions triggered by demand arrivals for item i: inventory of item i is depleted by one when each demand arrives. As seen in (3), there is neither a lower nor an upper limit for the inventory levels. Hence, the solutions are based on a truncated state space of $\boldsymbol{x} = (x_1, \dots, x_{|I|})$ such that the optimal production policy and the optimal long-run average cost of the system are unaffected.

Even though the developed dynamic programming (DP) formulation is presented under the discounted cost criterion, we derive numerical results under the average system cost criterion. This approach is adopted to circumvent the need to determine the discount parameter α and to render the results independent of the initial state. To compute the average system cost, we employ a value iteration algorithm on the DP formulation provided in equation (3), setting the discount rate to zero. At each stage, we calculate the value of the cost-to-go function and divide it by the completed number of stages. To stop the value iteration algorithm, a termination criterion is used. The algorithm continues to run until the absolute value of the difference between the average costs of all states is smaller than epsilon at any step. The algorithm is coded in MATLAB. Pseudo-code of the value iteration algorithm is given in Fig. 1 where *k* is for the current stage/step.

Here it should be noted that optimal production control requires a decision at each possible state of the system, and this poses a challenge in terms of computation time due to the nature of the value iteration algorithm. The number of equations increases exponentially in the number of products and production resources. As a result, we can solve the value iteration algorithm for small problem instances in reasonable times.

Assign an arbitrary value for J_0

$k = 0$

$While\ (difference > epsilon)$

$\quad k = k + 1$

$\quad\quad$ Repeat for each state:

$\quad\quad\quad$ Repeat for each possible production decision:

$\quad\quad\quad\quad$ Calculate $J_k^{cand}(J_{k-1}, state(u))$

$\quad\quad\quad J_k(state) = \min_u (J_k^{cand}(J_{k-1}, state(u)))$

$$difference = \max_{state \in SS} \max_{state' \in SS/\{state\}} \left| \frac{J_k\ (state)}{k} - \frac{J_k(state')}{k} \right|$$

$End\ while$

Fig. 1. Pseudo-code of the value iteration algorithm

In the following of this section, we examine the optimal policy structure for a multi-item MTS problem with backorders where $|I| = 2$, $s = 2$, $(\lambda_1, \lambda_2) = (1.40, 0.35)$, $(\mu_1, \mu_2) = (2.0, 0.5)$, $(h_1, h_2) = (1.0, 0.5)$, $(b_1, b_2) = (20, 10)$. When both production lines are idle, Table 3 presents a representation of all possible production decisions and Figure 2 illustrates optimal production policy. The optimal average system cost is calculated as 7.12. For the net inventory levels (6,6) in Fig. 2, we see that production ends and does not start when the inventory level of item 1 diminishes. At first glance, this might appear counterintuitive, as the stockout risk typically rises when the inventory level decreases for any product. However, this behavior is a consequence of the non-preemptive production assumption. In this context, both production lines remain idle and can be activated immediately when needed. For example, for the net inventory levels (3,6) in Figure 2, if a demand occurs for product 1, the optimal policy produces product 1 on both lines. However, if a demand occurs for product 2, the optimal policy produces product 1 on only one line. When the inventory of product 2 is slightly reduced, product 2 is also produced. If the inventory level of product 1 is low regardless of the inventory level of product 2, the optimal policy produces product 1 at both production resources. If the inventory level of product 1 is medium and the inventory level of product 2 is low, the optimal policy switches a resource to produce product 2. If the inventory level of product 1 is medium but the inventory level of product 2 is not low, the optimal policy turns off the production lines and keeps them idle to start the production immediately for any product if needed. Only if the inventory level for product 1 increases further, this option is given up and the production for product 2 starts at both resources.

The optimality of base stock policies is established solely for preemptive single-server MTS systems comprising two items, identical production rates, backorders, and discounted cost, as demonstrated by Ha (1997c). The findings of Ha (1997c) are expanded upon by De Vericourt et al. (2000), revealing that the Myopic (T) priority policy is optimal for states when the product with a smaller $b_i\mu_i$ index experiences backorders. As of now, there is no knowledge of additional implications regarding the structure of optimal production policies for the backordering case.

Table 3
Production control representation of the optimal policy

Decision	Produce nothing	Produce item 1 at both lines	Produce only item 1 at a time	Produce item 1 and item 2	Produce only item 2 at a line	Produce item 2 at both lines
Label	0	1	2	3	4	5
Color	(light grey)	(blue)	(orange)	(grey)	(yellow)	(green)

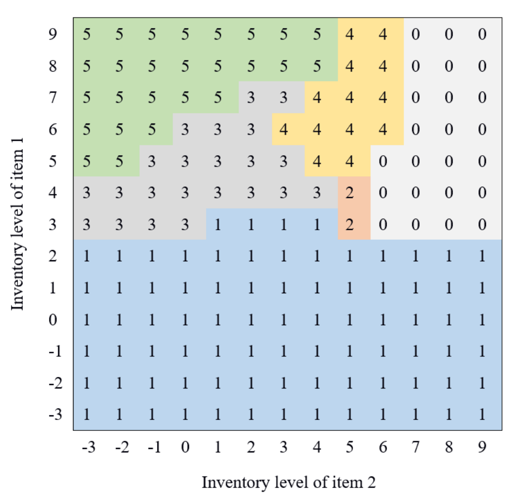

Fig. 2. Optimal production policy for backordering case

3.2. *Lost Sales Case*

In this section, we consider the same problem but this time shortages are assumed to be lost and lost sales cost c_i is incurred for each unsatisfied demand of product i. In this case, the state space is expressed as in (4). In (4), the lower bound of the state variable x_i is zero due to the lost sales assumption.

$$SS = \left\{ \left((x_1. \dots . x_{|I|}). (y_1. \dots . y_{|I|}) \right) | x_i \in \mathbb{Z}^+ \cup \{0\}. i \in I. \sum_{i=1}^{|I|} y_i \leq s \right\} \tag{4}$$

The structure of Eq. (2) written for the backordering case holds for the lost sales case. However, the cost-to-go function in Eq. (3) should be modified. The lost sales version of Equation (3) is given in Equation (5) where lost sales cost is incurred for the shortages. In $T_{D_i}(\boldsymbol{x}, \boldsymbol{u})$, if there is an inventory on hand $(x_i > 0)$, the inventory of item i is depleted by one unit when demand arrives, otherwise $(x_i = 0)$ it is lost, and the current state variables do not alter.

We again obtain the numerical results under the average system cost criterion to avoid determining discount parameter α and to make the results independent from the initial state. To calculate the average system cost, we apply the same value iteration algorithm whose pseudo-code is presented in Fig. 1.

$$J(\boldsymbol{x}.\boldsymbol{y}) = \frac{1}{v+\alpha} \min_{\substack{y_i \leq u_i \leq (s - \sum_{i' \neq i} y_{i'}) \\ \forall i}} \left\{ \sum_{i=1}^{|I|} h_i x_i + \left(s \max_i\{\mu_i\} - \sum_{i=1}^{|I|} u_i \mu_i \right) J(\boldsymbol{x}.\boldsymbol{u}) + \sum_{i=1}^{|I|} T_{P_i}(\boldsymbol{x}.\boldsymbol{u}) + \sum_{i=1}^{|I|} T_{D_i}(\boldsymbol{x}.\boldsymbol{u}) \right\} \tag{5}$$

where

$$T_{P_i}(\boldsymbol{x}, \boldsymbol{u}) = u_i \mu_i \min \left\{ J(\boldsymbol{x} + \boldsymbol{e_i}, \boldsymbol{u} - \boldsymbol{e_i}), \min_{i' \epsilon I} \{ \boldsymbol{x} + \boldsymbol{e_i}, \boldsymbol{u} - \boldsymbol{e_i} + \boldsymbol{e_{i'}} \} \right\}$$

$$T_{D_i}(\boldsymbol{x}, \boldsymbol{u}) = \begin{cases} \lambda_i J(\boldsymbol{x} - \boldsymbol{e_i}, \boldsymbol{u}) & x_i > 0 \\ c_i + J(\boldsymbol{x}, \boldsymbol{u}) & x_i = 0 \end{cases}$$

We now investigate the structure of the optimal production policy for a multi-item MTS problem with lost sales where $|I| = 2$, $s = 2$, $(\lambda_1, \lambda_2) = (1.40, 0.35)$, $(\mu_1, \mu_2) = (2.0, 0.5)$, $(h_1, h_2) = (1.0, 0.5)$, $(b_1, b_2) = (20, 10)$. Table 3 presents a representation of all possible production decisions and Figure 3 illustrates optimal production policy when all production lines are idle. The optimal average system cost is calculated as 5.47. As seen in Figure 3, the optimal production decisions behave as in the backordering case, but the inventory level terminates at the origin.

Inventory level of item 1 \ Inventory level of item 2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
14	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0
13	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0
12	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0
11	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0
10	5	5	4	0	0	0	0	0	0	0	0	0	0	0	0
9	5	5	4	0	0	0	0	0	0	0	0	0	0	0	0
8	5	5	4	0	0	0	0	0	0	0	0	0	0	0	0
7	5	5	4	0	0	0	0	0	0	0	0	0	0	0	0
6	5	4	4	0	0	0	0	0	0	0	0	0	0	0	0
5	4	4	4	0	0	0	0	0	0	0	0	0	0	0	0
4	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Fig. 3. Optimal production policy for lost sales case

In the context of lost sales multi-item MTS systems, various studies explore different combinations of idleness and scheduling policies, assessing their performance against the optimal one (Veatch and Wein, 1996). Similar to the backordering case, there is no existing knowledge of additional implications concerning the structure of optimal production policies for the lost sales scenario.

4. Proposed Heuristic Production Policies

In this section, we introduce new eight scheduling policies for the make-to-stock (MTS) systems with a multi-production resource for both backordering and lost sales cases. They are explained in detail in Sections 4.1 to 4.5.

As stated in the literature review section, where a single item is produced and production is made from only one source, the only decision is when to turn production on or off and the base stock policy (BSP) is optimal. Hence, in the multi-product version of that problem, considering the single production resource, heuristic production policies in the literature employ BSPs as idleness policies. However, if there exists a single product but multiple production resources at the same time, the optimal policy is a state-dependent base-stock type (Bulut & Fadıloğlu, 2011). As we extend the multi-item literature to a multi-resource case, we benefit from our pre-proposed policy for a single-item multi-production resource system. However, the BSP is near-optimal for a single-item multi-production resource system without start-up costs in many instances. Therefore, we again employ BSPs to determine when to start or stop production even in a multi-production resource case when the start-up costs are negligible.

If the inventory positions of all products are equal to the base stock levels (maximum inventory levels), production is not activated in any production resource. If only one product's inventory position is lower than its maximum stock level and the others are at their base stock levels, that product is selected to produce on an idle line. If there are at least two products whose inventory positions are lower than their base stock levels, one is selected to produce according to a scheduling rule, and then an idle production resource is activated to produce this selected product. If all production resources are busy, the new production starts after production is completed at a resource. Until the inventory positions of all products reach their maximum inventory levels, the selection rule repeats based on all the above-mentioned possible situations. Thus, according to the scheduling rule, the same product can be produced in more than one line at any time t.

Let $o^{\pi}(\boldsymbol{x},\boldsymbol{y}) \in \{1,2,\dots,|I|\}$ represent the selected product to produce by scheduling policy π when the inventory level vector is $\boldsymbol{x}$ and the vector of active production resources is $\boldsymbol{y}$. Therefore, we can write scheduling policy π as an index policy given in (6).

$$o^{\pi}(\boldsymbol{x}.\boldsymbol{y}) = arg\,\max_{i} G_i^{\pi}(\boldsymbol{x}.\boldsymbol{y}) \tag{6}$$

where the $G_i^{\pi}(\boldsymbol{x},\boldsymbol{y})$, $1 \le i \le |I|$, are index functions

4.1. Extension of $b\mu$ rule (B2)

The $b\mu$ rule (B1) is a static policy that chooses a product based on the number of backorders and index $b_i\mu_i$, where b_i is the backordering cost rate of product i and μ_i is the production rate of a product i.

We here extend the $b\mu$ rule by considering the number of active production resources for both backordering and lost sales cases. For the backordering case, we change the production rate μ_i with $(y_i+1)\mu_i$. Turning to the lost sales case, we multiply $(y_i+1)\mu_i$ with a cost rate for a stock-out, $c_i\lambda_i$. For the extended $b\mu$ rule (B2), the index functions $G_i(\boldsymbol{x},\boldsymbol{y})$ for the backordering and lost sales cases are given in Eq. (7) and Eq. (8), respectively.

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} 0 & if\ min(x_j) < 0\ and\ x_i \ge 0 \\ b_i(y_i+1)\mu_i & otherwise \end{cases} \tag{7}$$

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} 0 & if\ min(x_j) = 0\ and\ x_i > 0 \\ c_i\lambda_i(y_i+1)\mu_i & otherwise \end{cases} \tag{8}$$

4.2. Extension of $b\mu/h\mu$ rule (BH2)

In the $b\mu/h\mu$ rule (BH1), when at least two products have backorders, the product which has the biggest value of the index $b_i\mu_i$ is chosen (given priority) to produce where b_i is the backordering cost rate and μ_i is the production rate for product i. When all the products have no backorder, but their inventory levels are less than their base-stock (produce-up-to) levels, the product which has the smallest value of the index $h_i\mu_i$ is chosen to produce where h_i is the holding cost rate for product i.

We extend the $b\mu/h\mu$ rule (BH1) by considering the number of active production resources. For the backordering case, we change the production rate μ_i with $(y_i+1)\mu_i$. For the lost sales case, we employ the cost rate for a stock-out $c_i\lambda_i$ instead of b_i. For the extended $b\mu/h\mu$ rule (BH2), the index functions $G_i(\boldsymbol{x},\boldsymbol{y})$ for the backordering and lost sales cases are given in (9) and (10), respectively.

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} -h_i(y_i+1)\mu_i & if\ min(x_j) \geq 0\ and\ x_i \geq 0 \\ 0 & if\ min(x_j) < 0\ and\ x_i \geq 0 \\ b_i(y_i+1)\mu_i & x_i < 0 \end{cases} \tag{91}$$

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} -h_i(y_i+1)\mu_i & if\ min(x_j) > 0\ and\ x_i > 0 \\ 0 & if\ min(x_j) = 0\ and\ x_i > 0 \\ c_i\lambda_i(y_i+1)\mu_i & x_i = 0 \end{cases} \tag{10}$$

4.3. Extensions of switching rule (S2, S3, and S4)

In the switching rule (S1), it always awards higher priority to backordered products. Among the products that are backordered, priority is given to the one with the largest $b_i\mu_i$. If no product is backordered, priority is given to the one with the largest $b_i\mu_i(1-x_i/CL_i)$ where CL_i denotes the base stock level of product i and $(1-x_i/CL_i)$ denotes the proportion of unfilled base stock. The larger $(1-x_i/CL_i)$ is, the closer a product will be to being backordered.

We extend the switching rule (S1) in three ways. In S2, for the backordering case, we first change the production rate μ_i with $(y_i+1)\mu_i$ in the original switching rule. For the lost sales case, we multiply $(y_i+1)\mu_i$ with a cost rate for a stock-out, $c_i\lambda_i$. For the extended switching rule (S2), the index functions $G_i(\boldsymbol{x},\boldsymbol{y})$ for the backordering and lost sales cases are given in (11) and (12), respectively.

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} b_i(y_i+1)\mu_i(1-x_i/CL_i) & if\ min(x_j) \geq 0\ and\ x_i \geq 0 \\ 0 & if\ min(x_j) < 0\ and\ x_i \geq 0 \\ b_i(y_i+1)\mu_i & x_i < 0 \end{cases} \tag{11}$$

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} c_i\lambda_i(y_i+1)\mu_i(1-x_i/CL_i) & if\ min(x_j) > 0\ and\ x_i > 0 \\ 0 & if\ min(x_j) = 0\ and\ x_i > 0 \\ c_i\lambda_i(y_i+1)\mu_i & x_i = 0 \end{cases} \tag{12}$$

In the second alternative of the extended switching rule (S3), we again use the information of active production resources by changing the inventory level (x_i) with the inventory position (x_i+y_i). For S3, the index functions $G_i(\boldsymbol{x},\boldsymbol{y})$ for the backordering and lost sales cases are given in Eq. (13) and Eq. (14), respectively.

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} b_i\mu_i(1-(x_i+y_i)/CL_i) & if\ min(x_j) \geq 0\ and\ x_i \geq 0 \\ 0 & if\ min(x_j) < 0\ and\ x_i \geq 0 \\ b_i\mu_i & x_i < 0 \end{cases} \tag{13}$$

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} c_i\lambda_i\mu_i(1-(x_i+y_i)/CL_i) & if\ min(x_j) > 0\ and\ x_i > 0 \\ 0 & if\ min(x_j) = 0\ and\ x_i > 0 \\ c_i\lambda_i\mu_i & x_i = 0 \end{cases} \tag{14}$$

Finally, we employ our changes in S2 and S3 together in the third alternative of the extended switching rule (S4). For S4, the index functions $G_i(\boldsymbol{x},\boldsymbol{y})$ for the backordering and lost sales cases are given in (15) and (16), respectively.

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} b_i(y_i+1)\mu_i(1-(x_i+y_i)/CL_i) & if\ min(x_j) \geq 0\ and\ x_i \geq 0 \\ 0 & if\ min(x_j) < 0\ and\ x_i \geq 0 \\ b_i(y_i+1)\mu_i & x_i < 0 \end{cases} \tag{15}$$

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} c_i\lambda_i(y_i+1)\mu_i(1-(x_i+y_i)/CL_i) & if\ min(x_j) > 0\ and\ x_i > 0 \\ 0 & if\ min(x_j) = 0\ and\ x_i > 0 \\ c_i\lambda_i(y_i+1)\mu_i & x_i = 0 \end{cases} \tag{16}$$

4.4. Extensions of myopic rules (MP2 and MT2)

Myopic (P) policy (MP1) is related to the fully myopic policy of using the cost rate after one transition, which is the $b\mu/h\mu$ rule (BH1). In the BH1, if we produce product i, and ignore the effects of demands, the expected instantaneous rate of change in the cost is just h_i if $x_i \geq 0$ and $-b_i$ if $x_i < 0$. The policy chooses i to minimize this cost-change rate; it chooses the largest decrease if there are backorders and the smallest increase otherwise.

Myopic (P) proposed by Zipkin (1995), considers the expected cost rate after one production time. It can be viewed as a policy if we represent the resource by a Poisson process that is always on, with a decision made at the end of each processing time as to whether to load an item or run empty. According to the Myopic (P), if a product is currently being produced, the next product is selected to be produced when the production of that product is completed. Production cannot be interrupted; hence this policy has a non-preemptive nature. Myopic (P) looks ahead one production time and calculates the cost difference between having one more unit of product i and not having it. Then, it allocates the production capacity to the product that has the smallest cost increment (if on-hand inventory increases) or the biggest cost decrement (if the number of backorders decreases).

Let $P_i^1, P_i^2, \ldots, P_i^{y_i+1}$ be i.i.d. exponential random variables with rate μ_i. $D_i(P_{y_i}^{min})$ is the number of demands of product i in the interval [0,$P_{y_i}^{min}$] where $P_{y_i}^{min} = min(P_i^1, P_i^2, \ldots, P_i^{y_i+1})$ is exponential with a rate $(y_i+1)\mu_i$.

If the current inventory level is x_i and the number of active production resources is y_i, the expected cost rate of product i after a minimum of (y_i+1) production times is given by

$$g_i(x_i.y_i) = E[b_i[D_i(P_{y_i}^{min}) - x_i]^+ + h_i[x_i - D_i(P_{y_i}^{min})]^+] \tag{17}$$

If the current inventory level is (x_i+1), the expected cost rate of product i after minimum of production times is $g_i(x_i + 1, y_i)$. Thus, $\Delta g_i(x_i, y_i) = g_i(x_i+1, y_i) - g_i(x_i, y_i)$ is the rate at which having one more product i increases (or decreases) the instantaneous expected cost. The myopic heuristic then selects the product i with the maximum $-(y_i+1)\mu_i\Delta g_i(x_i, y_i)$. A simple calculation reveals that

$$\Delta g_i(x_i.y_i) = -b_iP(D_i(P_{y_i}^{min}) > x_i) + h_iP(D_i(P_{y_i}^{min}) \leq x_i) \tag{182}$$

Each $D_i(P_{y_i}^{min})$ has a geometric distribution with parameter $(1-p_i) = \lambda_i/(\lambda_i + (y_i+1)\mu_i)$. Then, for the backordering case, a formal definition of the function $G_i(\boldsymbol{x}, \boldsymbol{y})$ for the extended Myopic (P) rule (MP2) is given by

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} b_i(y_i+1)\mu_i - (b_i+h_i)(y_i+1)\mu_iP(D_i(P_{y_i}^{min}) \leq x_i) & if\ x_i \geq 0 \\ b_i(y_i+1)\mu_i & otherwise \end{cases} \tag{19}$$

where $P(D_i(P_{y_i}^{min}) \leq x_i) = 1 - (\lambda_i/(\lambda_i + (y_i+1)\mu_i))^{1+x_i}$

Now, consider $T_i^1, T_i^2, \ldots, T_i^{y_i+1}$ be i.i.d. exponential random variables with rate $(\mu_i - \lambda_i)$. $D_i(T_{y_i}^{min})$ is the number of demands of product i in the interval [0,$T_{y_i}^{min}$] where $T_{y_i}^{min} = min(T_i^1, T_i^2, \ldots, T_i^{y_i+1})$ is exponential with a rate $((y_i+1)\mu_i - \lambda_i)$. For the backordering case, the mathematical definition of the functions $G_i(\boldsymbol{x}, \boldsymbol{y})$ for the extended Myopic (T) policy (MT2) is given by

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} b_i(y_i+1)\mu_i - (b_i+h_i)(y_i+1)\mu_i P\left(D_i(T_{y_i}^{min}) \leq x_i\right) & if\ x_i \geq 0 \\ b_i(y_i+1)\mu_i & otherwise \end{cases} \tag{20}$$

where $P\left(D_i(T_{y_i}^{min}) \leq x_i\right) = 1 - (\lambda_i/((y_i+1)\mu_i)^{1+x_i}$

Turning to the lost sales case, the expected cost rate of product i after a minimum of (y_i+1) production times is given by

$$g_i(x_i.y_i) = E[c_i\lambda_i[D_i(P_{y_i}^{min}) - x_i]^+ + \ h_i[x_i - D_i(P_{y_i}^{min})]^+] \tag{21}$$

$$\Delta g_i(x_i.y_i) = -c_i\lambda_i p_i {q_i}^{x_i} + h_i P\left(D_i(P_{y_i}^{min}) \leq x_i\right) \tag{22}$$

where $p_i = (y_i+1)\mu_i/(\lambda_i + (y_i+1)\mu_i)$, $q_i = 1 - p_i$,

$P\left(D_i(P_{y_i}^{min}) \leq x_i\right) = 1 - {q_i}^{1+x_i}$

The mathematical definition of the functions $G_i(\boldsymbol{x}, \boldsymbol{y})$ for the extended Myopic (P) policy (MP2) for the lost sales case is given by

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} c_i\lambda_i p_i {q_i}^{x_i}(y_i+1)\mu_i - h_i P\left(D_i(P_{y_i}^{min}) \leq x_i\right) & if\ x_i \geq 0 \\ c_i\lambda_i p_i {q_i}^{x_i}(y_i+1)\mu_i & otherwise \end{cases} \tag{23}$$

where $p_i = (y_i+1)\mu_i/(\lambda_i + (y_i+1)\mu_i)$, $q_i = 1 - p_i$,

$P\left(D_i(P_{y_i}^{min}) \leq x_i\right) = 1 - {q_i}^{1+x_i}$

The mathematical definition of the functions $G_i(\boldsymbol{x}, \boldsymbol{y})$ for the extended Myopic (T) policy (MT2) for the lost sales case is given in

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \begin{cases} c_i\lambda_i p_i {q_i}^{x_i}(y_i+1)\mu_i - h_i P\left(D_i(T_{y_i}^{min}) \leq x_i\right) & if\ x_i \geq 0 \\ c_i\lambda_i p_i {q_i}^{x_i}(y_i+1)\mu_i & otherwise \end{cases} \tag{24}$$

where $p_i = ((y_i+1)\mu_i - \lambda_i)/(y_i+1)\mu_i$, $q_i = 1 - p_i$,

$P\left(D_i(T_{y_i}^{min}) \leq x_i\right) = 1 - {q_i}^{1+x_i}$

4.5. Extension of rolling horizon scheduling rule (RH2)

For the systems with two different products, the rolling horizon scheduling rule makes a choice about which product will be produced first by looking at the expected costs of schedules. The expected cost of a schedule is described as the total expected cost accrued during the duration required for producing both items. To illustrate this concept, consider a hypothetical system involving two products identified as i and j. If the expected cost associated with the schedule where product i is produced first is lower than the expected cost linked to the schedule where product j is produced first, prioritizing the production of product i might be more pressing than that of product j. Therefore, product i is chosen to produce first. According to the rule, after the production of product i is completed, the production of product j begins immediately in the production resource. Thus, the production schedule (i, j) is selected for this example.

For the multi-production resource case, the mathematical expression of $\Delta g_{ij}(x_i, y_i)$ is calculated as in Eq. (25) when shortages are backordered.

$$\Delta g_{ij}(x_i. y_i) = P(D_i(P_{y_i}^{min}) > x_i) b_i E[P_{y_j}^{min}] - \sum_{u=0}^{x_i}\sum_{w=0}^{x_i-u} P\big(D_i(P_{y_i}^{min}) = u\big) P\left(D_i(P_{y_j}^{min}) = w\right) E[P_{y_j}^{min}|w] h_i + \sum_{u=0}^{x_i}\sum_{w=x_i-u+1}^{\infty} P\big(D_i(P_{y_i}^{min}) = u\big) P\left(D_i(P_{y_j}^{min}) = w\right) E[P_{y_j}^{min}|w]\left(-h_i\left(\frac{x_i-u+1}{w+1}\right) + b_i\left(\frac{w-x_i+u}{w+1}\right)\right), \tag{25}$$

When shortages are lost, the mathematical expression of $\Delta g_{ij}(x_i, y_i)$ is calculated as in (26).

$$g_{ij}(x_i. y_i) = P(D_i(P_{y_i}^{min}) > x_i) c_i \lambda_i E[P_{y_j}^{min}] - \sum_{u=0}^{x_i}\sum_{w=0}^{x_i-u} P\big(D_i(P_{y_i}^{min}) = u\big) P\left(D_i(P_{y_j}^{min}) = w\right) E[P_{y_j}^{min}|w] h_i + \sum_{u=0}^{x_i}\sum_{w=x_i-u+1}^{\infty} P\big(D_i(P_{y_i}^{min}) = u\big) P\left(D_i(P_{y_j}^{min}) = w\right) E[P_{y_j}^{min}|w]\left(-h_i\left(\frac{x_i-u+1}{w+1}\right) + c_i\lambda_i\left(\frac{w-x_i+u}{w+1}\right)\right) \tag{26}$$

Considering the presence of Poisson arrivals occurring at a rate λ takes place over a time period distributed exponentially with rate $y\mu$ has a geometric distribution with parameter $p = 1/(1 + (\lambda/y\mu))$, we can compute above probabilities as $P\big(D_i(P_{y_i}^{min}) = u\big) = (1-p)^u p$, where $u \in \mathbb{N}_0$ and $p = 1/(1 + (\lambda_i/y_i\mu_i))$, and $P(D_i(P_{y_j}^{min}) = w)) = (1-p)^u p$ where $u \in \mathbb{N}_0$ and $p = 1/(1 + (\lambda_i/y_j\mu_j))$. It is also shown that $E[P_{y_j}^{min}|w] = (w+1)/(\lambda_i + y_j\mu_j)$. The mathematical representation of the index functions $G_i(\boldsymbol{x}, \boldsymbol{y})$ and $G_j(\boldsymbol{x}, \boldsymbol{y})$ for the rolling horizon scheduling rule (RH2) is given by

$$G_i(\boldsymbol{x}.\boldsymbol{y}) = \Delta g_{ij}(x_i. y_i)$$
$$G_j(\boldsymbol{x}.\boldsymbol{y}) = \Delta g_{ji}(x_j. y_i) \tag{27}$$

5. Numerical Study

In this section, we aim to measure the performance of the production policies we develop for multi-resource production systems, with numerical experiments. To achieve this goal, we compare those developed policies among themselves and with the optimal one, using test beds from the most relevant studies (Perez & Zipkin, 1997; Tiemessen et al., 2017). The numerical study contains problem instances with two products sharing the same production resources. In line with the literature, we consider identical and non-identical production rates to measure the impact of the difference in production rates on the performance of policies. Here, we consider three different combinations for the production rates vector $\boldsymbol{\mu} = (\mu_1, \mu_2)$: $\boldsymbol{\mu} = (\mu, \mu)$, $\boldsymbol{\mu} = (4\mu, \mu)$, and $\boldsymbol{\mu} = (\mu, 4\mu)$. The biggest distinction between our test environment and theirs: we allow multiple production resources.

For each problem instance, optimal long-run average costs are obtained by implementing the value iteration algorithm to the dynamic programming formulation. All results for the heuristic policies are obtained by computer simulation. They are modeled using ARENA Simulation Software. We determine the best base-stock levels through a direct search, evaluating each candidate policy by simulation. Our problem is a steady-state simulation model which has no natural termination time. However, we define a stopping criterion based on a certain number of events occurring in the system, revealing stable statistics. We consider a single replication and define a counter variable to keep a count of the number of times the production resource has seized an entity. When the terminating condition is satisfied (when the counter variable reaches a specified number), the replication is stopped. Lastly, the long-run average system cost is collected given the base stock level of each product. Fig. 4 shows an example of the average cost as a function of the counter variable. We start collecting a steady-state statistic after the counter variable reaches 8 million for the backordering case and 3 million for the lost sales cases, in which two-digit accuracy in the decimal part is obtained.

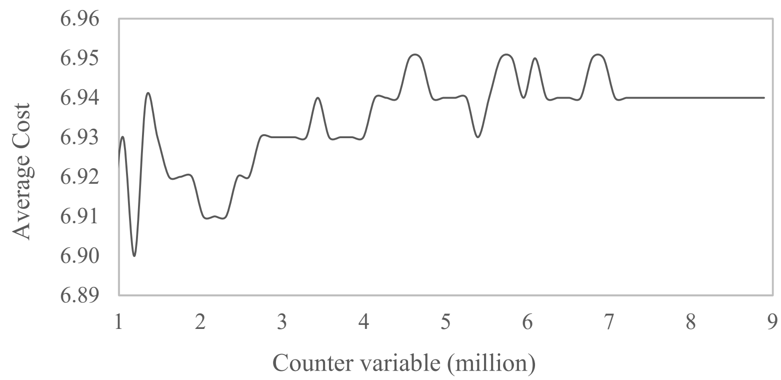

Fig. 4. Average cost as a function of event occurrence

This section reports the results of a numerical study comparing the heuristic policies and optimal policies. In our problem instances, we consider a production facility producing two different products. We assume that $\frac{\lambda_1}{s\mu_1} = \frac{\lambda_2}{s\mu_2} = \frac{\rho}{2}$, i.e., $\rho_1 = \rho_2 = \frac{\rho}{2}$. The inventory holding cost for product 1 consistently remains at one. The ratios b_1/h_1 (or c_1/h_1) and b_2/h_2 (or c_2/h_2) are equal to the common value b/h (or c/h), so cost asymmetry is measured by the ratio h_2/h_1. Product 2 always has the lowest costs, so $\frac{h_2}{h_1} \leq 1$. Three settings for the production rates vector $\boldsymbol{\mu}$ are examined: $i.\ \mu_1 = \mu_2$. $ii.\ \mu_1 = 4\mu_2$. $iii.\ \ 4\mu_1 = \mu_2$. Four values of s $(1,2,3,$ and $4)$, two values of ρ $(0.7$ and $0.6)$, three of h_2/h_1 $(0.9,\ 0.7,$ and $0.5)$, and two of b/h (or c/h) $(20$ and $80)$. In total, 288 problem instances are examined to measure the performance of the various production policies for both lost sales and backordering cases.

In the following of this chapter, for each s value, we provide the optimality gap tables when either production rates are identical or not. For the backordering case, the optimal expected average costs and The optimality gaps for the developed production policies are presented in Tables 4 – 6 when $s = 1$, in Tables 7 – 9 when $s = 2$, in Tables 10 – 12 when $s = 3$, and in Tables 13 – 15 when $s = 4$. For the lost sales case, both are shown in Tables 16 – 18 when $s = 1$, in Tables 19 – 21 when $s = 2$, in Tables 22 – 24 when $s = 3$, and in Tables 25 – 27 when $s = 4$.

For both backordering and lost sales cases, when the number of available production resources equals one and the production rates are identical, it is seen that the optimality gap of each heuristic policy is small. Specifically, the minimum optimality gaps of the switching rule (S1), myopic rules (MP1 and MT1), and rolling horizon rule (RH1) are no more than 1%, the average optimality gaps of these policies are less than 3%, and the maximum optimality gaps are at most 5%. When the production rates are non-identical for $s = 1$, the gaps are much higher. It is seen that while the ratio between the production rates alters, the optimality gaps increase for all heuristic policies. Nevertheless, S1 and RH1 show better performance compared to the others in this respect. Larger optimality gaps are observed in cases where the production time for the inexpensive product exceeds that of the expensive product, i.e., $\mu_1 = 4\mu_2$. This seems to be related to the non-preemptive nature.

When the number of available production resources is more than one and the production rates are identical, we see that the optimality gaps of B1, S3, MP2, MT2, and RH2 policies are small. While the average optimality gap remains below 4%, the maximum optimality gap does not exceed 8% for these policies. However, notable differences arise for non-identical production rates when $s > 1$. However, B1, S3, MT2, and RH2 show better performance in this respect.

6. Conclusion

This study considers a multi-item make-to-stock (MTS) system with parallel production lines for both lost sales and backordering environments. Inter-demand and processing times are assumed to be independent Exponential random variables. We model this system as an $M/M/s$ multi-item MTS queue and aim to minimize average inventory holding and shortage costs. For each item of the above-described system, the controller should decide when to (re)start and stop production on the lines. We formulate dynamic programming models for both backordering and lost sales cases, exploring the characteristics of optimal production policies through the outcomes of the value iteration algorithm. Additionally, we introduce near-optimal alternative production policies, comprising idleness and scheduling policies. These new policies extend the multi-server framework of those proposed in the literature for multi-item single-server MTS systems. Through an extensive numerical study, we assess and compare the performances of these extended heuristic policies. The experimental results demonstrate that the majority of the proposed production policies perform near-optimally in various cases.

This work might be extended by allowing non-identical production resources, batch arrivals, or batch processing settings. If the production resources are not identical, the number of active production resources should be tracked for each resource group. For batch demand and arrival cases, random events that occur would cause an increase or decrease in inventory more than once.

A compelling extension would be examining multi-item multi-production resource systems with general production times. In that case, the age or time-to-finish information of all ongoing production orders should be tracked. In order to yield approximations for the systems having general production times, multi-stage production systems can also be modeled using phase-type distributions such as Coxian and Erlang.

Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article.

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Appendix

Table 4

Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (1, 1, 1)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	8.61	5.53	5.53	7.04	7.04	4.37	4.37	4.37	0.00	0.00	0.00
0.7	0.9	80	12.26	7.36	7.36	11.32	11.32	4.77	4.77	4.77	0.00	0.00	0.00
0.7	0.7	20	7.53	3.39	3.39	6.92	6.92	3.12	3.12	3.12	0.00	0.00	0.00
0.7	0.7	80	10.75	5.12	5.12	13.15	13.15	2.74	2.74	2.74	0.00	0.00	0.00
0.7	0.5	20	6.43	0.75	0.75	7.08	7.08	0.59	0.59	0.59	1.70	0.00	0.00
0.7	0.5	80	9.14	3.21	3.21	16.84	16.84	2.04	2.04	2.04	2.60	0.00	0.00
0.6	0.9	20	6.39	3.66	3.66	3.18	3.18	2.79	2.79	2.79	0.00	0.00	0.00
0.6	0.9	80	9.01	7.11	7.11	9.62	9.62	5.04	5.04	5.04	0.00	0.00	0.00
0.6	0.7	20	5.64	2.85	2.85	3.00	3.00	2.13	2.13	2.13	0.00	0.00	0.00
0.6	0.7	80	8	4.81	4.81	10.23	10.23	3.37	3.37	3.37	0.26	0.00	0.00
0.6	0.5	20	4.87	2.22	2.22	3.16	3.16	2.22	2.22	2.22	0.00	0.00	0.00
0.6	0.5	80	6.96	2.27	2.27	10.96	10.96	1.51	1.51	1.51	0.07	0.00	0.00
			Minimum	0.75	0.75	3.00	3.00	0.59	0.59	0.59	0.00	0.00	0.00
			Average	4.02	4.02	8.54	8.54	2.89	2.89	2.89	0.39	0.00	0.00
			Maximum	7.36	7.36	16.84	16.84	5.04	5.04	5.04	2.60	0.00	0.00

Table 5

Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (1, 4, 1)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.40	7.30	7.30	31.46	31.46	7.30	7.30	7.30	29.59	23.54	6.50
0.7	0.9	80	13.46	6.53	6.53	69.84	69.84	8.13	8.13	8.13	40.64	29.73	7.33
0.7	0.7	20	8.38	9.21	9.21	36.31	36.31	9.21	9.21	9.21	34.77	27.00	8.41
0.7	0.7	80	12.01	8.54	8.54	77.78	77.78	9.00	9.00	9.00	48.56	35.05	8.20
0.7	0.5	20	7.33	12.11	12.11	43.08	43.08	12.11	12.11	12.11	41.51	33.57	11.31
0.7	0.5	80	10.53	11.42	11.42	88.43	88.43	11.42	11.42	11.42	57.84	42.07	10.73
0.6	0.9	20	7.15	7.59	7.59	23.03	23.03	7.59	7.59	7.59	22.11	22.67	7.09
0.6	0.9	80	10.37	9.72	9.72	43.94	43.94	10.89	10.89	10.89	35.61	29.72	10.39
0.6	0.7	20	6.43	9.39	9.39	25.97	25.97	9.39	9.39	9.39	26.14	26.14	8.89
0.6	0.7	80	9.32	11.77	11.77	50.53	50.53	11.77	11.77	11.77	37.53	30.10	11.27
0.6	0.5	20	5.68	12.24	12.24	30.32	30.32	12.24	12.24	12.24	31.16	27.90	11.74
0.6	0.5	80	8.25	14.04	14.04	59.15	59.15	14.04	14.04	14.04	43.77	36.52	13.70
			Minimum	6.53	6.53	23.03	23.03	7.30	7.30	7.30	22.11	22.67	6.50
			Average	9.99	9.99	48.32	48.32	10.26	10.26	10.26	37.44	30.34	9.63
			Maximum	14.04	14.04	88.43	88.43	14.04	14.04	14.04	57.84	42.07	13.70

Table 6
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (1, 1, 4)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.42	7.45	7.45	32.14	32.14	6.91	6.91	6.91	24.63	19.94	4.61
0.7	0.9	80	13.48	8.78	8.78	65.96	65.96	9.85	9.85	9.85	38.82	28.35	6.60
0.7	0.7	20	8.43	6.57	6.57	28.47	28.47	6.73	6.73	6.73	19.72	17.60	4.43
0.7	0.7	80	12.05	8.21	8.21	71.59	71.59	9.97	9.97	9.97	33.62	23.86	7.67
0.7	0.5	20	7.39	6.17	6.17	24.65	24.65	8.13	8.13	8.13	14.26	12.91	5.83
0.7	0.5	80	10.58	7.89	7.89	61.15	61.15	9.98	9.98	9.98	24.97	17.63	7.68
0.6	0.9	20	7.11	8.28	8.28	24.05	24.05	8.28	8.28	8.28	19.03	19.03	6.48
0.6	0.9	80	10.31	11.22	11.22	43.75	43.75	10.77	10.77	10.77	30.70	27.45	8.50
0.6	0.7	20	6.32	7.25	7.25	20.95	20.95	7.25	7.25	7.25	15.00	15.00	5.45
0.6	0.7	80	9.16	10.29	10.29	38.01	38.01	9.67	9.67	9.67	24.92	22.42	7.87
0.6	0.5	20	5.49	6.68	6.68	17.83	17.83	6.07	6.07	6.07	10.62	10.62	4.27
0.6	0.5	80	7.95	9.77	9.77	31.60	31.60	9.99	9.99	9.99	18.39	15.91	8.19
			Minimum	6.17	6.17	17.83	17.83	6.07	6.07	6.07	10.62	10.62	4.27
			Average	8.21	8.21	38.35	38.35	8.63	8.63	8.63	22.89	19.23	6.47
			Maximum	11.22	11.22	71.59	71.59	10.77	10.77	10.77	38.82	28.35	8.54

Table 7
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (2, 0.5, 0.5)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	8.77	5.63	13.23	7.12	11.87	13.23	4.18	11.71	1.37	1.71	1.32
0.7	0.9	80	12.46	7.81	21.65	9.73	17.12	19.02	4.53	18.12	1.28	1.10	1.23
0.7	0.7	20	7.76	3.35	16.20	5.95	14.82	15.76	2.33	14.65	1.47	2.04	1.42
0.7	0.7	80	10.99	5.11	25.39	10.92	20.55	22.36	3.30	20.15	1.91	1.82	1.86
0.7	0.5	20	6.62	2.34	2.34	6.25	3.63	2.40	1.77	2.08	0.68	2.05	0.63
0.7	0.5	80	9.39	2.98	2.98	14.04	8.50	3.84	1.88	2.64	1.44	1.38	1.39
0.6	0.9	20	6.58	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.9	80	9.30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.7	20	5.85	1.25	10.67	3.20	8.51	8.87	1.32	7.95	0.03	0.56	0.00
0.6	0.7	80	8.28	4.57	14.07	6.58	11.70	12.64	3.44	10.39	1.20	1.33	1.15
0.6	0.5	20	5.08	0.00	0.00	2.48	1.08	0.39	0.00	0.00	0.00	0.00	0.00
0.6	0.5	80	7.19	2.71	2.71	9.03	5.97	3.28	2.07	2.50	0.61	0.65	0.56
			Minimum	0.00	0.00	2.48	1.08	0.39	0.00	0.00	0.00	0.00	0.00
			Average	3.58	10.92	7.53	10.37	10.18	2.48	9.02	1.00	1.27	0.79
			Maximum	7.81	25.39	14.04	20.55	22.36	4.53	20.15	1.91	2.05	1.86

Table 8
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (2, 2, 0.5)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.23	9.34	9.34	31.74	31.74	10.86	9.34	10.91	20.20	10.24	8.80
0.7	0.9	80	13.11	10.12	10.12	71.66	71.66	14.12	10.77	11.16	30.85	10.57	10.39
0.7	0.7	20	8.18	11.64	11.64	38.17	38.17	13.44	11.64	13.17	27.00	12.74	11.03
0.7	0.7	80	11.63	12.62	12.62	80.41	80.41	15.23	13.29	15.92	33.89	12.95	12.86
0.7	0.5	20	7.12	14.78	14.78	45.70	45.70	16.21	14.78	14.78	32.50	12.99	14.07
0.7	0.5	80	10.1	16.11	16.11	92.67	92.67	19.47	16.11	17.99	38.72	14.37	15.61
0.6	0.9	20	7.05	7.83	7.83	18.00	18.00	11.29	7.83	7.83	16.04	9.56	7.12
0.6	0.9	80	10.05	11.34	11.34	43.15	43.15	15.86	11.47	13.06	22.02	10.90	10.98
0.6	0.7	20	6.30	10.17	10.17	21.68	21.68	10.17	10.17	10.17	18.79	11.14	9.38
0.6	0.7	80	8.98	13.46	13.46	48.73	48.73	15.67	13.46	16.24	28.32	11.74	12.91
0.6	0.5	20	5.54	13.36	13.36	25.11	25.11	13.36	13.36	13.36	21.05	13.00	12.45
0.6	0.5	80	7.88	16.59	16.59	56.19	56.19	19.44	16.59	16.59	33.88	16.27	15.95
			Minimum	7.83	7.83	18.00	18.00	10.17	7.83	7.83	16.04	9.56	7.12
			Average	12.28	12.28	47.77	47.77	14.59	12.40	13.43	26.94	12.20	11.80
			Maximum	16.59	16.59	92.67	92.67	19.47	16.59	17.99	38.72	16.27	15.95

Table 9
Optimality gap (%) for backordering case when $(\boldsymbol{s}, \boldsymbol{\mu_1}, \boldsymbol{\mu_2}) = (\mathbf{2}, \mathbf{0.5}, \mathbf{2})$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.28	8.21	8.21	28.47	28.47	10.98	8.34	9.80	15.87	8.67	6.72
0.7	0.9	80	13.21	10.38	10.38	69.56	69.56	12.80	10.30	11.67	23.38	10.43	9.17
0.7	0.7	20	8.34	7.40	7.40	24.51	24.51	11.69	7.70	9.06	12.49	8.91	5.90
0.7	0.7	80	11.88	9.52	9.52	61.34	61.34	15.22	9.20	11.98	15.93	7.15	7.94
0.7	0.5	20	7.34	7.25	32.32	20.42	35.80	25.86	8.54	24.02	14.58	11.47	6.50
0.7	0.5	80	10.5	8.96	49.46	51.75	71.83	27.28	7.59	24.23	12.06	12.67	6.16
0.6	0.9	20	7.06	7.83	7.83	16.08	16.08	10.21	7.83	7.83	13.03	9.33	5.71
0.6	0.9	80	10.06	11.66	11.66	37.33	37.33	14.13	9.65	10.35	17.54	12.09	8.16
0.6	0.7	20	6.31	6.74	6.74	13.45	13.45	8.76	6.74	6.74	9.59	8.76	4.36
0.6	0.7	80	8.97	10.59	10.59	32.43	32.43	13.62	7.37	9.59	13.07	10.48	5.70
0.6	0.5	20	5.53	5.91	24.68	10.74	28.84	22.39	4.32	22.48	12.48	11.95	1.61
0.6	0.5	80	7.84	9.66	30.63	26.62	37.76	22.97	8.11	17.87	11.72	10.60	6.51
			Minimum	5.91	6.74	10.74	13.45	8.76	4.32	6.74	9.59	7.15	1.61
			Average	8.68	17.45	32.73	38.12	16.33	7.97	13.80	14.31	10.21	6.20
			Maximum	11.66	49.46	69.56	71.83	27.28	10.30	24.23	23.38	11.66	9.17

Table 10
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (3, 0.33, 0.33)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.19	2.60	6.66	2.74	5.70	6.28	2.11	5.48	0.00	0.54	0.00
0.7	0.9	80	12.98	5.12	15.00	6.16	9.49	13.22	1.98	10.51	0.00	0.00	0.00
0.7	0.7	20	8.1	0.72	8.32	2.67	7.51	8.07	0.67	7.41	0.00	0.62	0.00
0.7	0.7	80	11.45	3.14	16.88	7.83	12.36	16.79	0.66	12.67	0.00	0.00	0.00
0.7	0.5	20	6.95	0.00	11.45	3.45	10.81	11.45	0.00	9.84	0.39	0.00	0.25
0.7	0.5	80	9.79	1.87	20.94	10.74	17.65	18.75	0.33	15.22	0.68	0.00	0.67
0.6	0.9	20	6.88	2.01	4.30	2.63	4.01	2.99	2.02	2.82	0.36	0.00	0.36
0.6	0.9	80	9.69	3.44	7.05	4.50	5.88	6.78	1.93	5.72	0.13	0.00	0.12
0.6	0.7	20	6.12	0.64	4.30	1.52	4.49	4.30	0.74	4.02	0.00	4.30	0.00
0.6	0.7	80	8.61	2.20	7.26	5.45	7.41	7.31	1.20	5.53	0.00	3.73	0.00
0.6	0.5	20	5.33	0.00	4.88	0.66	5.10	4.88	0.00	4.28	0.38	2.46	0.22
0.6	0.5	80	7.50	1.00	7.96	6.11	9.07	7.33	0.40	4.91	0.01	2.93	0.02
			Minimum	0.00	4.30	0.66	4.01	2.99	0.00	2.82	0.00	0.00	0.00
			Average	1.89	9.58	4.54	8.29	9.01	1.00	7.37	0.16	1.21	0.13
			Maximum	5.12	20.94	10.74	17.65	18.75	2.11	15.22	0.68	4.30	0.67

Table 11
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (3, 1.32, 0.33)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.54	6.87	6.87	23.31	23.31	11.14	7.32	8.32	12.64	5.62	5.57
0.7	0.9	80	13.54	8.92	8.92	58.99	58.99	13.69	6.97	9.62	14.13	6.99	6.94
0.7	0.7	20	8.46	9.46	9.46	28.29	28.29	9.18	9.46	9.70	14.88	6.69	6.64
0.7	0.7	80	11.99	10.85	10.85	68.29	68.29	15.89	8.77	10.65	19.98	8.34	8.29
0.7	0.5	20	7.35	12.94	12.94	35.27	35.27	13.86	12.94	12.94	19.05	8.68	8.63
0.7	0.5	80	10.41	13.67	13.67	80.88	80.88	16.62	13.67	11.91	20.76	11.15	11.10
0.6	0.9	20	7.27	5.53	5.53	9.08	9.08	7.99	5.53	5.53	9.82	5.32	5.27
0.6	0.9	80	10.27	9.13	9.13	29.84	29.84	11.45	7.80	9.19	16.41	7.80	7.75
0.6	0.7	20	6.50	7.08	7.08	11.40	11.40	8.62	7.08	7.08	13.54	5.80	5.75
0.6	0.7	80	9.17	10.59	10.59	35.10	35.10	13.83	10.59	11.56	16.80	8.63	8.58
0.6	0.5	20	5.70	9.61	9.61	14.95	14.95	11.26	9.61	9.61	12.98	7.47	7.42
0.6	0.5	80	8.04	12.87	12.87	42.34	42.34	17.08	12.87	14.17	19.30	12.11	12.06
			Minimum	5.53	5.53	9.08	9.08	7.99	5.53	5.53	9.82	5.32	5.27
			Average	9.79	9.79	36.48	36.48	12.55	9.38	10.02	15.86	7.88	7.83
			Maximum	13.67	13.67	80.88	80.88	17.08	13.67	14.17	20.76	12.11	12.06

Table 12
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (3, 0.33, 1.32)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.60	3.98	3.98	18.67	18.67	11.20	4.88	5.85	10.74	1.66	4.08
0.7	0.9	80	13.64	7.31	7.31	52.49	52.49	13.48	4.80	6.58	11.37	7.83	4.00
0.7	0.7	20	8.64	2.88	28.53	15.58	27.15	27.07	3.01	22.67	16.11	9.66	2.21
0.7	0.7	80	12.28	6.08	41.40	47.22	34.19	25.07	3.93	18.25	11.35	9.09	3.13
0.7	0.5	20	7.62	2.30	20.16	12.59	19.08	23.32	4.20	18.95	12.47	8.83	3.40
0.7	0.5	80	10.87	5.02	35.43	38.52	28.41	31.19	7.36	16.78	9.74	7.23	6.56
0.6	0.9	20	7.26	3.90	3.90	8.55	8.55	4.26	3.90	3.90	10.41	5.17	3.10
0.6	0.9	80	10.28	7.12	7.12	30.49	30.49	13.25	7.34	9.32	11.99	8.53	6.54
0.6	0.7	20	6.47	3.35	23.37	8.05	23.71	23.37	2.78	18.30	13.80	10.85	1.98
0.6	0.7	80	9.20	5.52	28.30	26.27	23.84	26.53	6.35	19.18	15.34	9.97	5.55
0.6	0.5	20	5.67	2.84	17.39	7.62	17.71	17.39	3.37	15.63	9.52	6.77	2.57
0.6	0.5	80	8.08	4.02	20.37	20.46	16.67	21.73	4.54	17.88	10.80	7.77	3.74
			Minimum	2.30	3.90	7.62	8.55	4.26	2.78	3.90	9.52	1.66	1.98
			Average	4.53	19.77	23.88	25.08	19.82	4.71	14.44	11.97	7.78	3.95
			Maximum	7.31	41.40	52.49	52.49	31.19	7.36	22.67	16.11	10.85	6.56

Table 13
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (4, 0.25, 0.25)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.12	3.30	8.42	4.57	7.64	8.29	2.65	7.11	0.55	3.63	0.49
0.7	0.9	80	12.84	4.60	14.22	5.33	11.02	13.22	1.58	10.28	0.00	3.69	0.00
0.7	0.7	20	8.07	1.91	10.74	3.85	10.01	10.45	1.54	9.54	1.04	3.26	0.98
0.7	0.7	80	11.37	2.45	16.85	6.70	13.47	15.38	1.04	12.58	0.00	3.24	0.00
0.7	0.5	20	6.95	1.12	6.91	3.94	8.24	7.12	0.88	5.64	1.50	2.78	1.44
0.7	0.5	80	9.76	1.07	12.02	10.04	13.59	12.55	0.97	9.48	0.12	4.45	0.06
0.6	0.9	20	6.95	2.30	3.19	2.22	3.27	3.19	2.19	3.05	0.40	1.08	0.34
0.6	0.9	80	9.74	4.48	7.94	5.74	7.43	7.76	3.94	6.47	1.75	3.84	1.69
0.6	0.7	20	6.2	1.31	3.61	1.42	3.68	3.61	1.24	3.42	0.92	1.18	0.86
0.6	0.7	80	8.68	3.04	8.62	5.94	8.13	7.93	2.75	7.02	1.67	3.31	1.61
0.6	0.5	20	5.41	0.78	2.11	1.15	2.51	2.13	0.78	2.11	1.13	1.66	1.07
0.6	0.5	80	7.57	1.86	5.02	6.02	8.08	5.20	1.65	4.07	1.40	2.99	1.34
			Minimum	0.78	2.11	1.15	2.51	2.13	0.78	2.11	0.00	1.08	0.00
			Average	2.35	8.30	4.74	8.09	8.07	1.77	6.73	0.87	2.92	0.82
			Maximum	4.60	16.85	10.04	13.59	15.38	3.94	12.58	1.75	4.45	1.61

Table 14
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (4, 1, 0.25)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.48	9.81	9.81	25.69	25.69	13.72	8.69	10.72	16.13	6.95	6.10
0.7	0.9	80	13.44	9.84	9.84	57.51	57.51	18.42	10.87	13.32	16.58	8.64	7.79
0.7	0.7	20	8.42	11.65	11.65	29.48	29.48	15.48	11.65	10.86	17.24	9.55	8.70
0.7	0.7	80	11.91	12.07	12.07	67.32	67.32	19.24	11.49	14.90	21.23	10.46	9.61
0.7	0.5	20	7.33	14.50	14.50	34.91	34.91	16.41	14.50	13.75	19.50	9.35	8.50
0.7	0.5	80	10.36	15.19	15.19	79.78	79.78	20.41	15.19	15.07	25.42	11.52	10.67
0.6	0.9	20	7.29	7.06	7.06	11.37	11.37	7.06	7.06	7.06	11.93	4.97	4.12
0.6	0.9	80	10.25	9.23	9.23	28.66	28.66	13.62	9.51	11.35	15.54	9.14	8.29
0.6	0.7	20	6.52	8.90	8.90	13.39	13.39	8.90	8.90	8.90	13.63	5.08	4.23
0.6	0.7	80	9.16	10.97	10.97	33.42	33.42	17.40	11.02	11.53	16.90	8.02	7.17
0.6	0.5	20	5.73	11.36	11.36	16.35	16.35	11.36	11.36	11.36	12.67	6.82	5.97
0.6	0.5	80	8.04	13.50	13.50	38.30	38.30	18.93	13.50	13.86	19.55	11.73	10.88
			Minimum	7.06	7.06	11.37	11.37	7.06	7.06	7.06	11.93	4.97	4.23
			Average	11.17	11.17	36.35	36.35	15.08	11.14	11.89	17.20	8.52	7.66
			Maximum	15.19	15.19	79.78	79.78	20.41	15.19	15.07	25.42	11.73	10.88

Table 15
Optimality gap (%) for backordering case when $(s, \mu_1, \mu_2) = (4, 0.25, 1)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	b/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	9.53	7.31	32.28	17.41	35.25	32.26	5.93	25.10	23.39	10.22	5.11
0.7	0.9	80	13.52	8.11	39.08	48.31	47.62	29.97	8.98	18.84	19.43	9.60	8.16
0.7	0.7	20	8.56	6.54	27.04	16.14	29.72	28.17	5.34	21.54	18.77	9.17	4.52
0.7	0.7	80	12.15	7.47	33.06	41.72	39.42	30.30	5.44	18.27	17.05	12.23	4.62
0.7	0.5	20	7.54	6.27	21.26	13.70	22.48	20.37	5.08	17.76	15.37	10.17	4.26
0.7	0.5	80	10.74	7.08	25.98	33.96	29.63	29.10	6.02	18.32	13.11	4.99	5.20
0.6	0.9	20	7.29	4.06	23.98	6.52	25.91	23.98	4.06	22.76	16.74	9.53	3.24
0.6	0.9	80	10.25	6.43	31.92	24.58	31.19	27.72	6.79	19.59	18.41	10.99	5.97
0.6	0.7	20	6.50	3.55	21.09	5.37	23.25	21.55	3.55	19.91	14.91	11.25	2.73
0.6	0.7	80	9.16	5.62	27.10	20.91	26.46	28.30	6.46	20.87	14.89	9.87	5.64
0.6	0.5	20	5.69	3.27	15.71	4.27	17.52	17.10	2.09	15.71	9.89	7.89	1.27
0.6	0.5	80	8.04	4.99	21.42	16.68	20.90	21.50	5.97	17.64	12.81	6.77	5.15
			Minimum	3.27	15.71	4.27	17.52	17.10	2.09	15.71	9.89	4.99	1.27
			Average	5.89	26.66	20.80	29.11	25.86	5.48	19.69	16.23	9.39	4.66
			Maximum	8.11	39.08	48.31	47.62	32.26	8.98	25.10	23.39	12.23	8.16

Table 16
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (1, 1, 1)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	4.72	4.53	4.53	4.30	4.30	4.53	4.53	4.53	3.60	3.60	4.28
0.7	0.9	80	7.55	1.10	1.10	2.11	2.11	0.05	0.05	0.05	1.72	1.72	0.00
0.7	0.7	20	4.21	4.49	4.49	3.78	3.78	4.49	4.49	4.49	3.23	3.23	4.24
0.7	0.7	80	6.71	0.69	0.69	3.89	3.89	0.19	0.19	0.19	0.24	0.24	0.00
0.7	0.5	20	3.70	4.41	4.41	3.11	3.11	4.41	4.41	4.41	4.41	4.41	4.16
0.7	0.5	80	5.87	0.15	0.15	6.18	6.18	0.31	0.31	0.31	0.31	0.31	0.06
0.6	0.9	20	4.16	2.38	2.38	2.28	2.28	2.38	2.38	2.38	1.78	1.78	2.13
0.6	0.9	80	6.63	2.22	2.22	1.63	1.63	3.63	3.63	3.63	1.66	1.66	3.38
0.6	0.7	20	3.72	2.45	2.45	2.10	2.10	2.45	2.45	2.45	1.69	1.69	2.20
0.6	0.7	80	5.91	2.52	2.52	0.61	0.61	3.59	3.59	3.59	3.20	3.20	3.34
0.6	0.5	20	3.28	2.53	2.53	1.86	1.86	2.53	2.53	2.53	2.53	2.53	2.28
0.6	0.5	80	5.18	2.70	2.70	0.91	0.91	2.70	2.70	2.70	2.59	2.59	2.45
			Minimum	0.15	0.15	0.61	0.61	0.05	0.05	0.05	0.24	0.24	0.00
			Average	2.51	2.51	2.73	2.73	2.61	2.61	2.61	2.25	2.25	2.38
			Maximum	4.53	4.53	6.18	6.18	4.53	4.53	4.53	4.41	4.41	4.28

Table 17
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (1, 4, 1)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.61	5.37	5.37	11.16	11.16	5.37	5.37	5.37	8.41	8.41	4.97
0.7	0.9	80	10.28	7.15	7.15	37.68	37.68	7.15	7.15	7.15	21.25	21.25	6.75
0.7	0.7	20	6.10	6.72	6.72	11.69	11.69	6.72	6.72	6.72	10.08	10.08	6.32
0.7	0.7	80	9.43	8.32	8.32	41.36	41.36	8.32	8.32	8.32	24.72	24.72	7.92
0.7	0.5	20	5.59	8.32	8.32	12.29	12.29	8.32	8.32	8.32	10.50	10.50	7.92
0.7	0.5	80	8.56	9.99	9.99	46.11	46.11	9.99	9.99	9.99	28.46	28.46	9.59
0.6	0.9	20	5.76	6.20	6.20	7.81	7.81	6.20	6.20	6.20	6.25	6.25	5.80
0.6	0.9	80	8.73	7.00	7.00	28.64	28.64	7.00	7.00	7.00	18.36	18.36	6.60
0.6	0.7	20	5.33	6.98	6.98	8.14	8.14	6.98	6.98	6.98	6.38	6.38	6.58
0.6	0.7	80	8.01	8.48	8.48	32.78	32.78	8.48	8.48	8.48	20.47	20.47	8.08
0.6	0.5	20	4.88	8.34	8.34	9.00	9.00	8.34	8.34	8.34	6.97	6.97	7.94
0.6	0.5	80	7.30	10.10	10.10	35.66	35.66	10.10	10.10	10.10	22.84	22.84	9.70
			Minimum	5.37	5.37	7.81	7.81	5.37	5.37	5.37	6.25	6.25	4.97
			Average	7.75	7.75	23.53	23.53	7.75	7.75	7.75	15.39	15.39	7.35
			Maximum	10.10	10.10	46.11	46.11	10.10	10.10	10.10	28.46	28.46	9.70

Table 18
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (1, 1, 4)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.41	2.81	2.81	7.58	7.58	2.81	2.81	2.81	5.96	5.96	2.41
0.7	0.9	80	10.05	5.66	5.66	32.47	32.47	5.66	5.66	5.66	16.83	16.83	5.26
0.7	0.7	20	5.53	2.08	2.08	8.34	8.34	2.08	2.08	2.08	4.79	4.79	1.68
0.7	0.7	80	8.73	5.04	5.04	37.08	37.08	5.04	5.04	5.04	16.03	16.03	4.64
0.7	0.5	20	4.65	1.05	1.05	9.35	9.35	1.05	1.05	1.05	3.20	3.20	0.65
0.7	0.5	80	7.39	4.48	4.48	32.41	32.41	4.48	4.48	4.48	12.45	12.45	4.08
0.6	0.9	20	5.59	4.35	4.35	4.54	4.54	4.35	4.35	4.35	4.54	4.54	3.95
0.6	0.9	80	8.53	5.23	5.23	25.71	25.71	5.23	5.23	5.23	16.18	16.18	4.83
0.6	0.7	20	4.83	3.54	3.54	4.64	4.64	3.54	3.54	3.54	4.64	4.64	3.14
0.6	0.7	80	7.40	4.32	4.32	23.01	23.01	4.32	4.32	4.32	13.81	13.81	3.92
0.6	0.5	20	4.05	16.02	16.02	25.56	25.56	16.02	16.02	16.02	18.49	18.49	15.62
0.6	0.5	80	6.26	3.26	3.26	19.54	19.54	3.26	3.26	3.26	10.72	10.72	2.86
			Minimum	1.05	1.05	4.54	4.54	1.05	1.05	1.05	3.20	3.20	0.65
			Average	4.82	4.82	19.19	19.19	4.82	4.82	4.82	10.64	10.64	4.42
			Maximum	16.02	16.02	37.08	37.08	16.02	16.02	16.02	18.49	18.49	15.62

Table 19
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (2, 0.5, 0.5)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	4.86	3.35	3.35	3.35	3.35	3.35	3.35	3.35	3.35	3.35	2.95
0.7	0.9	80	7.75	0.77	1.60	1.38	1.38	1.60	0.95	0.19	0.10	0.10	0.55
0.7	0.7	20	4.34	3.16	3.16	3.16	3.16	3.16	3.16	3.16	3.16	3.16	2.76
0.7	0.7	80	6.93	0.04	1.92	2.24	1.56	1.13	1.13	0.30	0.42	0.29	0.73
0.7	0.5	20	3.83	3.19	3.19	3.19	3.19	3.19	3.19	3.19	3.19	3.19	2.79
0.7	0.5	80	6.08	0.39	0.39	3.70	1.51	0.67	0.63	0.77	0.12	0.87	0.23
0.6	0.9	20	4.36	4.79	4.79	4.79	4.79	4.79	4.79	4.79	4.79	4.79	4.39
0.6	0.9	80	6.76	1.42	1.04	1.12	0.84	1.04	1.66	1.30	1.04	1.04	1.26
0.6	0.7	20	3.90	4.77	4.77	4.77	4.77	4.77	4.77	4.77	4.77	4.77	4.37
0.6	0.7	80	6.05	1.82	0.98	0.78	0.78	0.98	1.90	1.42	1.42	0.98	1.50
0.6	0.5	20	3.44	4.77	4.77	4.77	4.77	4.77	4.77	4.77	4.77	4.77	4.37
0.6	0.5	80	5.33	2.12	2.12	0.17	1.03	2.05	2.12	2.12	1.37	1.37	1.72
			Minimum	0.04	0.39	0.17	0.78	0.67	0.63	0.19	0.10	0.10	0.23
			Average	2.55	2.67	2.78	2.59	2.62	2.70	2.51	2.37	2.39	2.30
			Maximum	4.79	4.79	4.79	4.79	4.79	4.79	4.79	4.79	4.79	4.39

Table 20
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (2, 2, 0.5)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.55	6.52	6.52	7.44	7.44	6.52	6.52	6.52	4.72	2.37	6.02
0.7	0.9	80	10.03	8.69	8.69	29.67	29.67	8.69	8.69	8.69	14.29	7.74	8.19
0.7	0.7	20	6.02	7.03	7.03	7.03	7.03	7.03	7.03	7.03	6.00	3.34	6.53
0.7	0.7	80	9.15	10.52	10.52	30.39	30.39	10.52	10.52	10.52	15.26	8.30	10.02
0.7	0.5	20	5.47	5.01	5.01	5.01	5.01	5.01	5.01	5.01	4.66	3.13	4.51
0.7	0.5	80	8.24	13.16	13.16	31.74	31.74	13.16	13.16	13.16	19.04	9.61	12.66
0.6	0.9	20	5.69	5.62	5.62	6.22	6.22	5.62	5.62	5.62	4.09	4.09	5.12
0.6	0.9	80	8.56	7.09	7.09	20.63	20.63	7.09	7.09	7.09	9.68	5.27	6.59
0.6	0.7	20	5.25	4.74	4.74	4.74	4.74	4.74	4.74	4.74	4.29	2.69	4.24
0.6	0.7	80	7.80	8.99	8.99	21.71	21.71	8.99	8.99	8.99	11.46	6.86	8.49
0.6	0.5	20	4.75	3.56	3.56	3.56	3.56	3.56	3.56	3.56	3.03	3.03	3.06
0.6	0.5	80	7.04	11.28	11.28	23.00	23.00	11.28	11.28	11.28	12.50	7.67	10.78
			Minimum	3.56	3.56	3.56	3.56	3.56	3.56	3.56	3.03	2.37	3.06
			Average	7.68	7.68	15.93	15.93	7.68	7.68	7.68	9.08	5.34	7.18
			Maximum	13.16	13.16	31.74	31.74	13.16	13.16	13.16	19.04	9.61	12.66

Table 21
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (2, 0.5, 2)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.40	0.41	0.41	1.77	1.77	0.41	0.41	0.41	0.41	0.41	0.31
0.7	0.9	80	9.85	4.77	4.77	23.36	23.36	4.77	4.77	4.77	10.15	5.24	5.14
0.7	0.7	20	5.55	0.00	0.00	1.19	1.19	0.00	0.00	0.00	0.00	0.00	0.00
0.7	0.7	80	8.60	3.88	3.88	25.19	25.19	3.88	3.88	3.88	8.33	4.28	4.18
0.7	0.5	20	4.65	0.28	0.28	1.25	1.25	0.28	0.28	0.28	0.28	0.28	0.18
0.7	0.5	80	7.33	2.97	2.97	21.21	21.21	3.33	2.97	2.97	8.14	4.08	3.98
0.6	0.9	20	5.55	1.91	1.91	2.34	2.34	1.91	1.91	1.91	1.91	1.91	1.81
0.6	0.9	80	8.41	3.02	3.02	14.46	14.46	3.02	3.02	3.02	6.30	3.02	2.92
0.6	0.7	20	4.83	0.99	0.99	1.39	1.39	0.99	0.99	0.99	0.99	0.99	0.00
0.6	0.7	80	7.36	1.93	1.93	14.04	14.04	1.93	1.93	1.93	6.96	2.32	2.22
0.6	0.5	20	4.08	0.49	0.49	0.83	0.83	0.49	0.49	0.49	0.49	0.49	0.39
0.6	0.5	80	6.31	0.48	0.48	9.68	9.68	0.48	0.48	0.48	4.28	1.54	1.44
			Minimum	0.00	0.00	0.83	0.83	0.00	0.00	0.00	0.00	0.00	0.00
			Average	1.76	1.76	9.73	9.73	1.79	1.76	1.76	4.02	2.05	1.88
			Maximum	4.77	4.77	25.19	25.19	4.77	4.77	4.77	10.15	5.24	5.14

Table 22
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (3, 0.33, 0.33)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	4.98	0.18	0.18	0.16	0.16	0.18	0.18	0.18	0.34	0.34	0.24
0.7	0.9	80	7.94	0.23	0.09	0.40	0.57	0.57	0.73	0.58	0.13	0.13	0.03
0.7	0.7	20	4.45	0.27	0.27	0.31	0.31	0.27	0.27	0.27	0.49	0.49	0.39
0.7	0.7	80	7.08	0.44	0.00	1.29	1.26	0.30	0.61	0.42	0.30	0.30	0.20
0.7	0.5	20	3.92	0.38	0.38	0.54	0.54	0.38	0.38	0.38	0.38	0.71	0.61
0.7	0.5	80	6.21	0.53	0.29	2.58	2.29	0.29	0.47	0.53	0.21	0.98	0.88
0.6	0.9	20	4.52	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.09
0.6	0.9	80	6.94	0.59	0.50	0.75	0.97	0.20	0.20	0.17	0.53	0.53	0.43
0.6	0.7	20	4.05	1.33	1.33	1.33	1.33	1.33	1.33	1.33	1.33	1.33	0.00
0.6	0.7	80	6.20	0.52	0.55	1.13	1.26	0.29	0.26	0.24	0.73	0.73	0.63
0.6	0.5	20	3.57	1.23	1.23	1.23	1.23	1.23	1.23	1.23	1.23	1.23	1.13
0.6	0.5	80	5.46	0.42	0.59	1.61	1.61	0.59	0.42	0.38	0.31	0.57	0.47
			Minimum	0.18	0.00	0.16	0.16	0.18	0.18	0.17	0.13	0.13	0.00
			Average	0.61	0.55	1.04	1.06	0.57	0.61	0.58	0.60	0.71	0.51
			Maximum	1.33	1.33	2.58	2.29	1.33	1.33	1.33	1.33	1.33	1.13

Table 23
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (3, 1.32, 0.33)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.63	2.11	2.11	2.26	2.26	2.11	2.11	2.11	1.58	1.01	0.71
0.7	0.9	80	10.11	8.89	8.89	20.94	20.94	8.89	8.89	8.89	12.22	14.91	14.61
0.7	0.7	20	6.07	2.60	2.60	2.82	2.82	2.60	2.60	2.60	1.32	1.32	1.02
0.7	0.7	80	9.22	10.41	10.41	23.42	23.42	10.41	10.41	10.41	12.39	15.38	11.58
0.7	0.5	20	5.50	3.40	3.40	3.67	3.67	3.40	3.40	3.40	1.87	1.87	1.57
0.7	0.5	80	8.31	12.25	12.25	23.31	23.31	12.25	12.25	12.25	14.30	17.18	16.18
0.6	0.9	20	5.83	0.86	0.86	1.03	1.03	0.86	0.86	0.86	0.48	0.38	0.08
0.6	0.9	80	8.63	7.08	7.08	12.07	12.07	7.08	7.08	7.08	7.98	8.46	8.16
0.6	0.7	20	5.35	1.18	1.18	1.36	1.36	1.18	1.18	1.18	0.77	0.58	0.28
0.6	0.7	80	7.88	8.20	8.20	13.82	13.82	8.20	8.20	8.20	9.10	9.72	9.42
0.6	0.5	20	4.86	1.77	1.77	1.98	1.98	1.77	1.77	1.77	1.34	1.05	0.75
0.6	0.5	80	7.10	10.01	10.01	16.44	16.44	10.01	10.01	10.01	8.89	11.41	11.11
			Minimum	0.86	0.86	1.03	1.03	0.86	0.86	0.86	0.48	0.38	0.08
			Average	5.73	5.73	10.26	10.26	5.73	5.73	5.73	6.02	6.94	6.29
			Maximum	12.25	12.25	23.42	23.42	12.25	12.25	12.25	14.30	17.18	16.18

Table 24
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (3, 0.33, 1.32)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.48	1.03	1.03	1.25	1.25	1.03	1.03	1.03	1.03	1.03	0.93
0.7	0.9	80	9.94	3.65	3.65	13.45	13.45	4.11	3.65	3.65	8.80	9.00	8.70
0.7	0.7	20	5.63	1.26	1.26	1.42	1.42	1.26	1.26	1.26	1.26	1.26	1.16
0.7	0.7	80	8.71	2.72	2.72	12.34	9.94	3.35	2.72	2.72	6.83	9.06	6.73
0.7	0.5	20	4.77	1.78	1.78	1.87	1.87	1.78	1.78	1.78	1.78	1.78	1.68
0.7	0.5	80	7.43	2.18	2.18	11.63	7.47	2.54	2.18	2.18	6.30	7.91	6.20
0.6	0.9	20	5.71	0.16	0.16	0.35	0.35	0.16	0.16	0.16	0.16	0.16	0.06
0.6	0.9	80	8.47	4.43	4.43	7.10	7.10	4.43	4.43	4.43	4.57	4.43	4.47
0.6	0.7	20	4.97	0.04	0.04	0.22	0.22	0.04	0.04	0.04	0.04	0.04	0.00
0.6	0.7	80	7.41	3.78	3.78	6.17	6.17	4.04	3.78	3.78	4.05	3.82	3.95
0.6	0.5	20	4.22	0.12	0.12	0.26	0.26	0.12	0.12	0.12	0.12	0.12	0.02
0.6	0.5	80	6.34	3.08	3.08	5.08	5.08	3.61	3.08	3.08	3.50	3.68	3.40
			Minimum	0.04	0.04	0.22	0.22	0.04	0.04	0.04	0.04	0.04	0.00
			Average	2.02	2.02	5.09	4.55	2.21	2.02	2.02	3.20	3.52	3.11
			Maximum	4.43	4.43	13.45	13.45	4.43	4.43	4.43	8.80	9.06	8.70

Table 25
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (4, 0.25, 0.25)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	5.10	0.43	0.43	0.43	0.43	0.43	0.43	0.43	0.31	0.31	0.16
0.7	0.9	80	8.08	1.72	2.02	2.00	2.57	2.02	1.35	1.32	1.61	1.79	1.46
0.7	0.7	20	4.56	0.37	0.37	0.37	0.37	0.37	0.37	0.37	0.37	0.20	0.22
0.7	0.7	80	7.22	1.63	2.19	2.31	2.74	1.91	1.36	1.41	1.51	1.87	1.36
0.7	0.5	20	4.02	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.15
0.7	0.5	80	6.34	1.61	1.62	3.04	3.04	1.69	1.70	1.42	1.91	2.16	1.76
0.6	0.9	20	4.69	0.43	0.43	0.43	0.43	0.43	0.43	0.43	0.53	0.53	0.38
0.6	0.9	80	7.09	0.48	0.51	0.37	0.25	0.51	0.47	0.55	0.51	0.17	0.36
0.6	0.7	20	4.20	0.36	0.36	0.36	0.36	0.36	0.36	0.36	0.36	0.36	0.21
0.6	0.7	80	6.35	0.71	0.58	0.33	0.31	0.58	0.65	0.65	0.65	0.58	0.50
0.6	0.5	20	3.70	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.36
0.6	0.5	80	5.59	0.64	0.64	0.05	0.05	0.55	0.64	0.64	0.47	0.47	0.32
			Minimum	0.30	0.30	0.05	0.05	0.30	0.30	0.30	0.30	0.17	0.15
			Average	0.77	0.83	0.88	0.95	0.81	0.71	0.70	0.75	0.77	0.60
			Maximum	1.72	2.19	3.04	3.04	2.02	1.70	1.42	1.91	2.16	1.76

Table 26
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (4, 1, 0.25)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.71	3.11	3.11	3.93	3.93	3.11	3.11	3.11	2.56	2.15	1.65
0.7	0.9	80	10.21	7.82	7.82	15.85	15.85	8.19	7.82	7.82	9.03	7.62	7.12
0.7	0.7	20	6.15	3.76	3.76	3.76	3.76	3.76	3.76	3.76	2.78	2.18	1.68
0.7	0.7	80	9.31	9.05	9.05	14.73	14.73	9.05	9.05	9.05	10.38	7.87	7.37
0.7	0.5	20	5.59	3.26	3.26	3.26	3.26	3.26	3.26	3.26	2.88	2.86	2.36
0.7	0.5	80	8.4	11.14	11.14	15.62	15.62	11.14	11.14	11.14	11.74	9.73	9.23
0.6	0.9	20	5.94	1.40	1.40	1.40	1.40	1.40	1.40	1.40	1.18	1.18	0.68
0.6	0.9	80	8.76	5.70	5.70	8.54	8.54	5.70	5.70	5.70	4.93	4.86	4.36
0.6	0.7	20	5.44	1.19	1.19	1.19	1.19	1.19	1.19	1.19	0.99	0.99	0.49
0.6	0.7	80	7.99	6.86	6.86	8.56	8.56	6.86	6.86	6.86	5.88	5.43	4.93
0.6	0.5	20	4.94	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.75	0.75	0.25
0.6	0.5	80	7.21	7.88	7.88	8.75	8.75	7.88	7.88	7.88	6.46	5.95	5.45
			Minimum	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.75	0.75	0.25
			Average	5.17	5.17	7.21	7.21	5.21	5.17	5.17	4.96	4.30	3.80
			Maximum	11.14	11.14	15.85	15.85	11.14	11.14	11.14	11.74	9.73	9.23

Table 27
Optimality gap (%) for lost sales case when $(s, \mu_1, \mu_2) = (4, 0.25, 1)$

Parameters			Optimal Average Cost	Gap (%)									
ρ	h_2/h_1	c/h		B1	B2	BH1	BH2	S2	S3	S4	MP2	MT2	RH2
0.7	0.9	20	6.56	1.34	1.34	1.88	1.88	1.34	1.34	1.34	1.34	1.34	0.84
0.7	0.9	80	10.04	3.18	3.18	11.11	11.11	3.18	3.18	3.18	4.41	3.18	2.68
0.7	0.7	20	5.71	0.96	0.96	1.40	1.40	0.96	0.96	0.96	0.96	0.96	0.46
0.7	0.7	80	8.78	2.81	2.81	9.31	9.31	2.81	2.81	2.81	3.69	2.81	2.31
0.7	0.5	20	4.85	0.68	0.68	0.97	0.97	0.68	0.68	0.68	0.68	0.68	0.18
0.7	0.5	80	7.50	2.61	2.61	7.19	7.19	2.61	2.61	2.61	3.35	2.61	2.11
0.6	0.9	20	5.81	1.43	1.43	1.43	1.43	1.43	1.43	1.43	1.43	1.43	0.93
0.6	0.9	80	8.62	1.77	1.77	5.58	5.58	1.77	1.77	1.77	2.16	1.77	1.27
0.6	0.7	20	5.06	1.68	1.68	1.92	1.92	1.68	1.68	1.68	1.68	1.68	1.18
0.6	0.7	80	7.54	1.39	1.39	4.56	4.56	1.39	1.39	1.39	1.66	1.39	0.89
0.6	0.5	20	4.31	1.65	1.65	1.97	1.97	1.65	1.65	1.65	1.65	1.65	1.15
0.6	0.5	80	6.45	1.04	1.04	3.35	3.35	1.04	1.04	1.04	1.53	1.04	0.54
			Minimum	0.68	0.68	0.97	0.97	0.68	0.68	0.68	0.68	0.68	0.18
			Average	3.18	3.18	11.11	11.11	3.18	3.18	3.18	4.41	3.18	1.21
			Maximum	3.18	3.18	11.11	11.11	3.18	3.18	3.18	4.41	3.18	2.68