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Optimal pricing and inventory policies for non-instantaneous deteriorating items with permissible delay in payment: Fuzzy expected value model

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ABSTRACT

This study investigates optimal pricing and inventory policies for non-instantaneous deteriorating items with permissible delay in payment. The demand rate is as known, continuous and differentiable function of price while holding cost rate, interest paid rate and interest earned rate are characterized as independent fuzzy variables rather than fuzzy numbers as in previous studies. Under these general assumptions, we first formulated a fuzzy expected value model (EVM) and then some useful theoretical results have been derived to characterize the optimal solutions. An efficient algorithm is designed to determine the optimal pricing and inventory policy for the proposed model. The algorithmic procedure is demonstrated by means of numerical examples.

1. Introduction

According to the modern view, uncertainty is considered essential to science; it is not only an unavoidable phenomenon but has, in fact, a great utility in real world applications. In essence, uncertainty occurs not only due to a lack of information but also as a result of ambiguity (impreciseness) due to the semantic statements by experts. In context of the inventory management, experts usually make interval-valued or linguistic statements about the time parameters and relevant data of inventory system. These interval-valued or linguistic statements lead to non-stochastic uncertainties. The fuzzy set theory was developed to model uncertainties in non-stochastic sense.

During last two decades, several researchers have investigated various types of inventory problems in fuzzy environments to model uncertainties in non-stochastic sense (e.g. Park; 1987, Chen et al.; 1996, Roy and Maiti; 1997, Chang and Yao; 1998, Lee and Yao; 1999, Kao and Hsu; 2002, Chen and Ouyang; 2006, De & Goswami, 2006; Roy et al.; 2008). In aforementioned studies, the common feature is that the parameters (demand, cost coefficients etc.) were assumed to be triangular fuzzy numbers or trapezoidal fuzzy numbers. From literature survey, there are few literatures considered the parameters to be fuzzy variables. For instance, Wang et al. (2007) constructed EVM for EOQ model

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without backordering by characterizing the holding cost and ordering cost as fuzzy variables. Wang and Tang (2009) considered EVM for the EPQ problem with backorder in which the setup cost, the holding cost and the backorder cost are characterized as fuzzy variables, respectively. Recently, Soni and Shah (2011) developed fuzzy expected value production model by characterizing demand and production preparation time as fuzzy variables.

In recent years, researchers studied inventory problems for non-instantaneous items under different conditions. For example, Ouyang et al. (2006) studied an inventory model for non-instantaneous deteriorating items with permissible delay in payments. Geetha and Uthayakumar (2010) extended Ouyang et al.'s model incorporating time-dependent backlogging rate. However, both models consider constant demand rate and cost minimization objective. The assumption of constant demand is quite impractical in reality. It would be more realistic to consider the demand as selling price dependent. The basic idea is that price setting will influence the demand and potential profit. Therefore, we consider demand to be price sensitive.

Based on above discussion, we consider the time parameters, the holding cost rate and interest paid/earned rate in Geetha and Uthayakumar (2010) model may be varied slightly owing to some uncertainties in non-stochastic sense or uncontrolled environments. In addition, instead of constant demand rate we have assumed the demand rate as known, continuous and differentiable function of price. By incorporating above concepts we solve the new inventory model in the fuzzy sense. The main purpose of this study is to extend the paper of Geetha and Uthayakumar (2010) with a view to make the model more relevant and applicable practically.

The rest of the paper is organized as follows: In Section 2, the assumptions and notations which are used throughout the article are presented. In Section 3, fuzzy expected value model to maximize the total profit is formulated. Solution methodology comprising some useful theoretical results and algorithm to find the optimal solution is carried out in Section 4. Numerical examples are provided in Section 5 to illustrate the theory and the solution procedure. Finally, we draw a conclusion in Section 6.

2. Assumptions and Notations

The following notations and assumptions have been used in developing the mathematical model in this article.

2.1 Notations

A	:	The ordering cost per order.
M	:	Trade credit period.
c	:	The purchasing cost per unit.
p	:	The selling price per unit ($p > c$).
$\tilde{h}$	:	The inventory holding cost rate excluding interest charges rate which is imprecise in nature.
c_s	:	Unit shortage cost per unit time.
c_l	:	The cost of lost sale per unit.
$\tilde{i}_p$	:	The interest paid per dollar per unit time which is imprecise in nature.
$\tilde{i}_e$	:	The interest earned per dollar per unit time which is imprecise in nature.
Q	:	The order quantity.
t_1	:	The length of time in which the product has no deterioration.
t_2	:	The length of time in which the inventory has no shortage.
t_3	:	The length of period during which shortages are allowed.
T	:	Length of replenishment cycle, hence $T = t_2 + t_3$.
θ	:	The deterioration rate of the on-hand inventory over [t_1, t_2].

$\Pi(p, t_2, t_3)$: The total profit per unit time of inventory system.

$E(.)$: Expected value of $(.)$

2.2 Assumptions

(1) The inventory system involves single non-instantaneous deteriorating item.
(2) Demand rate $D(p)$ is any non-negative, continuous, decreasing function of the selling price.
(3) During the fixed period, t_1, the product has no deterioration. After that the on-hand inventory deteriorate with constant rate θ, where 0 < θ < 1. For simplicity, we assume that t_1 is given constant and $t_1 \le t_2$.
(4) There is no replacement or repair of deteriorated units during the period under consideration.
(5) Shortages are allowed and backlogged partially. We assume the fraction of shortages backorder is $1/(1+\delta x)$, where x is the waiting time up to the next replenishment and δ is backlogging parameter $0 \le \delta \le 1$. This function has been utilized by many researchers (e.g. Abad (1996, 2001), Dye (2007), Geetha and Uthayakumar (2010)).
(6) During the trade credit period, M, the account is not settled; the revenue is deposited in an interest bearing account. At the end of the period, the retailer pays off the item ordered, and starts to pay the interest charged on the item in stock.
(7) Replenishment rate is infinite and lead time is zero.
(8) The system operates for an infinite planning horizon.
(9) Holding cost rate, interest paid rate and interest earned rate are imprecise in nature and assumed to be non-interactive fuzzy variables defined on credibility space $(X_i, P(X_i), \mathrm{Cr}_i), i = 1,2,3$.

3. Model Formulation

3.1 The crisp inventory model

The inventory system evolves as follows: Q_1 units of items arrive at the inventory system at the beginning of each cycle. The inventory level is declining only due to demand rate over time interval [0, t_1]. The inventory level is reducing to zero owing to demand and deterioration during the time interval [t_1, t_2]. After that, inventory level becomes zero and shortages begin to be accumulated during [t_2, T]. The process is repeated as mentioned above.

Based on above description, the status of inventory at any instant of time $t \in [0, T]$ is governed by differential equation

$$\frac{dI(t)}{dt} = \begin{cases} -D(p) & 0 \le t \le t_1 \\ -\theta I(t) - D(p) & t_1 \le t \le t_2 \\ -D(p)/(1+\delta(T-t)) & t_2 \le t \le T \end{cases} \tag{1}$$

with boundary condition $I(0) = Q_1$ and $I(t_2) = 0$. The solution of Eq. (1) is

$$\frac{dI(t)}{dt} = \begin{cases} I_1(t) & \text{for } 0 \le t \le t_1 \\ I_2(t) & \text{for } t_1 \le t \le t_2 \\ I_3(t) & \text{for } t_2 \le t \le T \end{cases} \tag{2}$$

where

$$I_1(t) = \frac{D(p)}{\theta}\left[e^{\theta(t_2-t_1)} - \theta(t-t_1) - 1\right]$$

$$I_2(t)=\frac{D(p)}{\theta}\left[e^{\theta(t_2-t)}-1\right]$$

$$I_3(t)=\frac{-D(p)}{\delta}\left[\ln\left(1+\delta\left(T-t_2\right)\right)-\ln\left(1+\delta\left(T-t\right)\right)\right]$$

Also, the ordering quantity over the replenishment cycle can be determined as

$$Q=I_1(0)-I_3(T)=\frac{D(p)}{\theta}\left[e^{\theta(t_2-t_1)}-1+\theta t_1\right]+\frac{D(p)\ln\left(1+\delta t_3\right)}{\delta} \tag{3}$$

The profit of the inventory system consists of the following components.

1. The ordering cost (C_o) is A.
2. The inventory holding cost (C_h) per cycle is given by

$$C_h=h\int_0^{t_1}I_1(t)\,dt+h\int_{t_1}^{t_2}I_2(t)\,dt=hD(p)\left[\frac{t_1}{\theta}\left(e^{\theta(t_2-t_1)}-1\right)+\frac{t_1^2}{2}+\frac{1}{\theta^2}\left(e^{\theta(t_2-t_1)}-\theta\left(t_2-t_1\right)-1\right)\right]$$

3. The shortage cost (C_s) per cycle due to backlog is given by

$$C_s=c_s\int_{t_2}^{T}-I_3(t)\,dt=\frac{c_sD(p)}{\delta^2}\left[\delta\left(T-t_2\right)-\ln\left(1+\delta\left(T-t_2\right)\right)\right]=\frac{c_sD(p)}{\delta^2}\left[\delta t_3-\ln\left(1+\delta t_3\right)\right]$$

4. The opportunity cost (C_l) due to lost sale per cycle is given by

$$C_l=c_lD(p)\int_{t_2}^{T}\left[1-\frac{1}{1+\delta\left(T-t\right)}\right]dt=\frac{c_lD(p)}{\delta}\left[\delta t_3-\ln\left(1+\delta t_3\right)\right]$$

5. The purchase cost (C_p) is given by

$$C_p=c\times Q=\frac{cD(p)}{\theta}\left[e^{\theta(t_2-t_1)}-\theta\left(t_2-t_1\right)-1\right]-\frac{cD(p)}{\delta}\left[\delta t_3-\ln\left(1+\delta t_3\right)\right]+cD(p)\left(t_2+t_3\right)$$

6. The sale revenue (R) is given by

$$R=p\times\left[\int_0^{t_2}D(p)\,dt+\int_{t_2}^{T}D(p)/\left(1+\delta\left(T-t\right)\right)dt\right]=pD(p)\left(t_2+t_3\right)-\frac{pD(p)}{\delta}\left[\delta t_3-\ln\left(1+\delta t_3\right)\right]$$

Next, based on the parameter values t_1, t_2 and M, there are three cases to be explored.

Case 1: $0<M\le t_1$

Interest paid $=IP_1=ci_p\int_M^{t_1}I_1(t)dt+ci_p\int_{t_1}^{t_2}I_2(t)dt$

$$=ci_pD(p)\left[\frac{\left(t_1-M\right)}{\theta}\left(e^{\theta(t_2-t_1)}-1\right)+\frac{\left(t_1-M\right)^2}{2}+\frac{1}{\theta^2}\left(e^{\theta(t_2-t_1)}-\theta\left(t_2-t_1\right)-1\right)\right]$$

Interest earned $=IE_1=pi_e\int_0^{M}D(p)\,t\,dt=\dfrac{pi_eD(p)M^2}{2}$

Case 2: $t_1<M\le t_2$

Interest paid $= IP_2 = ci_p \int_M^{t_2} I_2(t)dt = \dfrac{ci_p D(p)}{\theta^2}\left(e^{\theta(t_2-M)} - \theta(t_2-M) - 1\right)$

Interest earned $= IE_2 = pi_e \int_0^M D(p)tdt = \dfrac{pi_e D(p) M^2}{2}$

Case 3: $M > t_2$

Interest paid $= IP_3 = 0$

Interest earned $= IE_3 = pi_e\left[\int_0^{t_2} D(p)tdt + (M-t_2)D(p)t_2\right] = pi_e\left[D(p)t_2(M-t_2/2)\right]$

Hence, the total profit per unit time for each case is

$$\Pi_i(p,t_2,t_3) = \frac{1}{t_2+t_3}\left[R - \left(C_o + C_h + C_s + C_l + C_p + IP_i - IE_i\right)\right], i = 1,2,3 \tag{4}$$

Hence our crisp problem is

$$\max \ \Pi(p,t_2,t_3) = \begin{cases} \Pi_1(p,t_2,t_3) & \text{if } 0 < M \le t_1 \\ \Pi_2(p,t_2,t_3) & \text{if } t_1 < M \le t_2 \\ \Pi_3(p,t_2,t_3) & \text{if } M > t_2 \end{cases}$$

where $\Pi_i(p,t_2,t_3), i = 1,2,3$ is given by Eq. (4).

3.2 Fuzzy Expected Value inventory model

In this article, we have considered the holding cost rate, interest paid rate and interest earned rate as fuzzy variables to tackle the reality in more effective way. When the parameters $\tilde{h}$, $\tilde{i}_p$ and $\tilde{i}_e$ (as per assumption) treated as fuzzy variables, the above inventory expressions become fuzzy and thereby the total profit per unit time becomes fuzzy variable on the credibility space $(X, P(X), \mathrm{Cr})$. If the decision maker wants to determine optimal pricing and inventory policy such that fuzzy expected value of the total profit is maximal, a fuzzy EVM can be constructed as follows,

$$\max \ E\left[\Pi(p,t_2,t_3)\right] = \begin{cases} E\left[\Pi_1(p,t_2,t_3)\right] & \text{if } 0 < M \le t_1 \\ E\left[\Pi_2(p,t_2,t_3)\right] & \text{if } t_1 < M \le t_2 \\ E\left[\Pi_3(p,t_2,t_3)\right] & \text{if } M > t_2 \end{cases} \tag{5}$$

where $E\left[\Pi_i(p,t_2,t_3)\right] = E\left[\dfrac{1}{t_2+t_3}\left\{R - \left(C_o + \widetilde{C_h} + C_s + C_l + C_p + \widetilde{IP_i} - \widetilde{IE_i}\right)\right\}\right], i = 1,2,3$

Next section carried out the solution methodology for fuzzy EVM along with theoretical results to identify global optimal solution for (p,t_2,t_3).

4. Solution Methodology

Using linearity of operator E the fuzzy EVM given by Eq. (5) can be reduced to following single objective crisp problem.

$$\max \ E\left[\Pi\left(p,t_2,t_3\right)\right]=\begin{cases} E\left[\Pi_1\left(p,t_2,t_3\right)\right] & \text{if } 0<M\le t_1 \\ E\left[\Pi_2\left(p,t_2,t_3\right)\right] & \text{if } t_1<M\le t_2 \\ E\left[\Pi_3\left(p,t_2,t_3\right)\right] & \text{if } M>t_2 \end{cases} \tag{6}$$

where $E\left[\Pi_i\left(p,t_2,t_3\right)\right]=\dfrac{1}{t_2+t_3}\left[R-\left(C_o+E\left[\tilde{C}_h\right]+C_s+C_l+C_p+E\left[\tilde{I}P_i\right]-E\left[\tilde{I}E_i\right]\right)\right]$

for $i=1,2,3$.

Case 1: $0<M\le t_1$

From Eq. (6), the expected value of the total profit during the replenishment cycle per unit time can be written as follows,

$$\begin{aligned} E\left[\Pi_1\left(p,t_2,t_3\right)\right]=\left(p-c\right)D\left(p\right)-\frac{D\left(p\right)}{\left(t_2+t_3\right)}&\left[\left\{\frac{\left(e^{\theta\left(t_2-t_1\right)}-\theta\left(t_2-t_1\right)-1\right)}{\theta^2}\left(E\left[\tilde{h}\right]+c\left(\theta+E\left[\tilde{i}_p\right]\right)\right)\right.\right. \\ &+\frac{E\left[\tilde{h}\right]t_1^2}{2}+\frac{cE\left[\tilde{i}_p\right]\left(t_1-M\right)^2}{2}+\frac{E\left[\tilde{h}\right]t_1+cE\left[\tilde{i}_p\right]\left(t_1-M\right)}{\theta}\left(e^{\theta\left(t_2-t_1\right)}-1\right) \\ &\left.\left.+\frac{\left(s+\delta\left(p-c+l\right)\right)}{\delta^2}\left(\delta t_3-\ln\left(1+\delta t_3\right)\right)+\frac{A}{D\left(p\right)}\right\}-\frac{pE\left[\tilde{i}_e\right]M^2}{2}\right]. \end{aligned} \tag{7}$$

To maximize the expected total profit per unit time, it is necessary to solve the following equations simultaneously.

$$\frac{\partial E\left[\Pi_1\left(p,t_2,t_3\right)\right]}{\partial t_2}=\frac{1}{\left(t_2+t_3\right)}\left[\Pi_1^{(1)}\left(p,t_2,t_3\right)-D\left(p\right)\left\{\frac{U_1\left(e^{\theta\left(t_2-t_1\right)}-1\right)}{\theta}+U_2e^{\theta\left(t_2-t_1\right)}\right\}\right]=0 \tag{8}$$

$$\frac{\partial E\left[\Pi_1\left(p,t_2,t_3\right)\right]}{\partial t_3}=\frac{1}{\left(t_2+t_3\right)}\left[\Pi_1^{(1)}\left(p,t_2,t_3\right)-D\left(p\right)\left\{\frac{V\left(p\right)t_3}{1+\delta t_3}\right\}\right]=0 \tag{9}$$

$$\begin{aligned} \frac{\partial E\left[\Pi_1\left(p,t_2,t_3\right)\right]}{\partial p}=&\left[D\left(p\right)+D'\left(p\right)\left(p-c\right)\right]\left[1-\frac{\delta t_3-\ln\left(1+\delta t_3\right)}{\delta\left(t_2+t_3\right)}+\frac{E\left[\tilde{i}_e\right]M^2}{2\left(t_2+t_3\right)}\right] \\ &-\frac{D'\left(p\right)}{\left(t_2+t_3\right)}\left[\frac{\left(e^{\theta\left(t_2-t_1\right)}-\theta\left(t_2-t_1\right)-1\right)U_1}{\theta^2}+\frac{E\left[\tilde{h}\right]t_1^2}{2}+\frac{cE\left[\tilde{i}_p\right]\left(t_1-M\right)^2}{2}\right. \\ &\left.+\frac{\left(e^{\theta\left(t_2-t_1\right)}-1\right)U_2}{\theta}+\frac{\left(s+\delta l\right)}{\delta^2}\left(\delta t_3-\ln\left(1+\delta t_3\right)\right)+\frac{A}{D\left(p\right)}-\frac{cE\left[\tilde{i}_e\right]M^2}{2}\right]=0 \end{aligned} \tag{10}$$

where $U_1 = E\left[\tilde{h}\right] + c\left(\theta + E\left[\tilde{i}_p\right]\right), U_2 = E\left[\tilde{h}\right]t_1 + cE\left[\tilde{i}_p\right]\left(t_1 - M\right)$, $V\left(p\right) = s + \delta\left(p - c + l\right)$

and $\Pi_1^{(1)}\left(p, t_2, t_3\right) = \left(p - c\right)D\left(p\right) - E\left[\Pi_1\left(p, t_2, t_3\right)\right]$.

In order to identify optimal solution for $\left(p, t_2, t_3\right)$, firstly we prove that for any given p, the optimal pair of values $\left(t_2, t_3\right)$ not only exists but also is unique. Once this is done, we shall derive the existence of p for optimal pair of values $\left(t_2, t_3\right)$.

From Eqs. (8) and (9) we obtain respectively,

$$\Pi_1^{(1)}\left(p, t_2, t_3\right) = D\left(p\right)\left\{\frac{U_1\left(e^{\theta(t_2 - t_1)} - 1\right)}{\theta} + U_2 e^{\theta(t_2 - t_1)}\right\}, \tag{11}$$

$$\Pi_1^{(1)}\left(p, t_2, t_3\right) = D\left(p\right)\left\{\frac{V\left(p\right)t_3}{1 + \delta t_3}\right\}. \tag{12}$$

Equating right hand side of Eqs. (11) and (12) we have

$$\frac{U_1\left(e^{\theta(t_2 - t_1)} - 1\right)}{\theta} + U_2 e^{\theta(t_2 - t_1)} = \frac{V\left(p\right)t_3}{1 + \delta t_3}. \tag{13}$$

For convenience, let $K_1\left(t_2\right)$ denote the left hand side of Eq. (13), that is,

$$K_1\left(t_2\right) = \frac{U_1\left(e^{\theta(t_2 - t_1)} - 1\right)}{\theta} + U_2 e^{\theta(t_2 - t_1)},$$

which implies

$$t_3 = \frac{K_1\left(t_2\right)}{V\left(p\right) - \delta K_1\left(t_2\right)} \tag{14}$$

Thus, t_3 is a function of t_2 and p.

Now, we substitute $\Pi_1^{(1)}\left(p, t_2, t_3\right)$ into Eq. (11) and making some algebraic manipulation, we obtain

$$\begin{aligned} &\frac{D\left(p\right)}{\left(t_2 + t_3\right)}\left\{\frac{U_1\left(e^{\theta(t_2 - t_1)} - 1\right)\left(1 - \theta\left(t_2 + t_3\right)\right)}{\theta^2} - \frac{U_1\left(t_2 - t_1\right)}{\theta} + \frac{E\left[\tilde{h}\right]t_1^2}{2} + \frac{cE\left[\tilde{i}_p\right]\left(t_1 - M\right)^2}{2}\right. \\ &\left. + \frac{U_2 e^{\theta(t_2 - t_1)}\left(1 - \theta\left(t_2 + t_3\right)\right)}{\theta} - \frac{U_2}{\theta} + \frac{V\left(p\right)\left(\delta t_3 - \ln\left(1 + \delta t_3\right)\right)}{\delta^2} + \frac{A}{D\left(p\right)} - \frac{pE\left[\tilde{i}_p\right]M^2}{2}\right\} = 0 \end{aligned} \tag{15}$$

Motivated by Eq. (15), we assume an auxiliary function, say $F_1\left(t_2\right), t_2 \in \left[t_1, \infty\right)$, where

$$F_1(t_2) = \frac{U_1\left(e^{\theta(t_2-t_1)}-1\right)\left(1-\theta(t_2+t_3)\right)}{\theta^2} - \frac{U_1(t_2-t_1)}{\theta} + \frac{E[\tilde{h}]t_1^2}{2} + \frac{cE[\tilde{i}_p](t_1-M)^2}{2}$$
$$+\frac{U_2 e^{\theta(t_2-t_1)}\left(1-\theta(t_2+t_3)\right)}{\theta} - \frac{U_2}{\theta} + \frac{V(p)\left(\delta t_3 - \ln(1+\delta t_3)\right)}{\delta^2} + \frac{A}{D(p)} - \frac{pE[\tilde{i}_p]M^2}{2} \tag{16}$$

and t_3 is given as in Eq. (14). Differentiating $F_1(t_2)$ with respect to t_2, and using the relation in Eqs. (13) and (14) we get

$$\frac{dF_1(t_2)}{dt_2} = -(U_1+\theta U_2)e^{\theta(t_2-t_1)}(t_2+t_3) < 0 \tag{17}$$

Thus, $F_1(t_2)$ is strictly decreasing function with respect to $t_2 \in [t_1,\infty)$ and it can be shown that as t_2 gets larger $F_1(t_2)$ approaches to $-\infty$. Moreover,

$$F_1(t_1) = \frac{E[\tilde{h}]t_1^2}{2} + \frac{cE[\tilde{i}_p](t_1-M)^2}{2} + \frac{U_2\left(1-\theta(t_1+t_3^{'})\right)}{\theta} - \frac{U_2}{\theta} + \frac{V(p)\left(\delta t_3^{'} - \ln(1+\delta t_3^{'})\right)}{\delta^2} + \frac{A}{D(p)} - \frac{pE[\tilde{i}_p]M^2}{2}$$

where $t_3^{'} = \dfrac{U_2}{\theta V(p) - U_2}$

Now, the optimal value of t_2 depends on sign of $F_1(t_1)$ so we examine two sub-cases as follows:

Sub-case 1.1: Let $F_1(t_1) \geq 0$. As $F_1(t_2)$ is strictly decreasing function of $t_2 \in [t_1,\infty)$, using intermediate value theorem, there exists unique value of t_2 (say $t_{2.1} \in [t_1,\infty)$) such that $F_1(t_{2.1}) = 0$ i.e. $t_{2.1}$ is the unique solution of Eq. (8). The corresponding value of $t_{3.1}$ can be found from Eq. (14).

Sub-case 1.2: If $F_1(t_1) < 0$ then $F_1(t_1)$. Since $F_1(t_2)$ is strictly decreasing function of $t_2 \in [t_1,\infty)$ and $\partial E[\Pi_1(p,t_2,t_3)]/\partial t_2 < 0, \forall t_2 \in [t_1,\infty)$. Hence, optimal value occurs at point $t_2 = t_1$ and corresponding optimal value of t_3 can be found from Eq. (14) and is given by $\dfrac{U_2}{\theta V(p) - U_2}$.

Summarizing the above arguments, we obtain the following result.

Lemma 4.1: For known p, we have

(a) If $F_1(t_1) \geq 0$ then there exist unique pair of values $(t_2,t_3) = (t_{2.1},t_{3.1})$ which satisfies Eqs. (8-9).

(b) If $F_1(t_1) < 0$ then the optimal value occurs at point $(t_2,t_3) = \left(t_1, \dfrac{U_2}{\theta V(p) - U_2}\right)$.

Suppose $(t_{2.1}^*, t_{3.1}^*)$ denotes the optimal value of (t_2,t_3) then we can obtain following result.

Theorem 4.1: For known p, the expected value of total profit function $E[\Pi_1(p,t_2,t_3)]$ is concave and attains its global maximum at point $(t_2,t_3) = (t_{2.1}^*, t_{3.1}^*)$.

Proof: From lemma 4.1 the pair of values $\left(t_{2.1}^*, t_{3.1}^*\right)$ which maximizes expected profit function $E\left[\Pi_1\left(p, t_2, t_3\right)\right]$ for any p is given by

$$\left(t_{2.1}^*, t_{3.1}^*\right)=\begin{cases}\left(t_{2.1}, t_{3.1}\right), & \text{if } \Delta(A, p) \geq 0 \\ \left(t_1, \dfrac{U_2}{\theta V(p)-U_2}\right), & \text{if } \Delta(A, p)<0\end{cases} \tag{18}$$

From Eqs. (8), (15) and (16) we have

$$\frac{\partial E\left[\Pi_1\left(p, t_2, t_3\right)\right]}{\partial t_2}=\frac{D(p) F_1\left(t_2\right)}{\left(t_2+t_3\right)} \tag{19}$$

At point $\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)$

$$\left[\frac{\partial^2 E\left[\Pi_1\left(p, t_2, t_3\right)\right]}{\partial t_2^2}\right]_{\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)}=-\left(U_1+\theta U_2\right) e^{\theta\left(t_{2.1}^*-t_1\right)}\left(t_{2.1}^*+t_{3.1}^*\right)<0$$

Furthermore, we can obtain from Eq. (9)

$$\left[\frac{\partial^2 E\left[\Pi_1\left(p, t_2, t_3\right)\right]}{\partial t_3^2}\right]_{\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)}=-\frac{1}{\left(t_{2.1}^*+t_{3.1}^*\right)^2}\left[\Pi_1^{(1)}\left(p, t_{2.1}^*, t_{3.1}^*\right)-D(p)\left(\frac{V(p) t_{3.1}^*}{1+\delta t_{3.1}^*}\right)\right]$$

$$+\frac{1}{\left(t_{2.1}^*+t_{3.1}^*\right)}\left[\left.\frac{\partial \Pi_1^{(1)}\left(p, t_2, t_3\right)}{\partial t_3}\right|_{\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)}-\frac{D(p) V(p)}{\left(1+\delta t_{3.1}^*\right)^2}\right]$$

$$=\frac{1}{\left(t_{2.1}^*+t_{3.1}^*\right)}\left[\left.\frac{\partial E\left[\Pi_1\left(p, t_2, t_3\right)\right]}{\partial t_3}\right|_{\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)}-\frac{D(p) V(p)}{\left(1+\delta t_{3.1}^*\right)^2}\right]=-\frac{D(p) V(p)}{\left(t_{2.1}^*+t_{3.1}^*\right)\left(1+\delta t_{3.1}^*\right)^2}<0$$

and $\left[\dfrac{\partial^2 E\left[\Pi_1\left(p, t_2, t_3\right)\right]}{\partial t_2 \partial t_3}\right]_{\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)}=0$. Moreover, the determinant of the Hessian matrix at point $\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)$ gives

$$|H|=\frac{D(p)\left(U_1+\theta U_2\right) V(p) e^{\theta\left(t_{2.1}^*-t_1\right)}}{\left(1+\delta t_{3.1}^*\right)^2}>0$$

Thus, the Hessian matrix is positive definite at point $\left(t_2, t_3\right)=\left(t_{2.1}^*, t_{3.1}^*\right)$. Hence, the pair of values $\left(t_{2.1}^*, t_{3.1}^*\right)$ gives global maximum of the optimization problem (5). This completes the proof.

Next, we analyze the condition under which the optimal selling price also exists and is unique. For the pair of values $(t_2, t_3) = (t_{2.1}^*, t_{3.1}^*)$ the necessary condition for $E\left[\Pi_1\left(p, t_{2.1}^*, t_{3.1}^*\right)\right]$ to be maximum is $\partial E\left[\Pi_1\left(p, t_{2.1}^*, t_{3.1}^*\right)\right] / \partial p = 0$. That is,

$$\frac{\partial E\left[\Pi_1\left(p,t_{2.1}^*,t_{3.1}^*\right)\right]}{\partial p} = \left[D(p)+D'(p)(p-c)\right]\left[1-\frac{\delta t_{3.1}^*-\ln\left(1+\delta t_{3.1}^*\right)}{\delta\left(t_{2.1}^*+t_{3.1}^*\right)}+\frac{E\left[\tilde{i}_e\right]M^2}{2\left(t_{2.1}^*+t_{3.1}^*\right)}\right]$$

$$-\frac{D'(p)}{\left(t_{2.1}^*+t_{3.1}^*\right)}\left[\frac{\left(e^{\theta\left(t_{2.1}^*-t_1\right)}-\theta\left(t_{2.1}^*-t_1\right)-1\right)U_1}{\theta^2}+\frac{E\left[\tilde{h}\right]t_1^2}{2}+\frac{cE\left[\tilde{i}_p\right]\left(t_1-M\right)^2}{2}\right. \tag{20}$$

$$\left.+\frac{\left(e^{\theta\left(t_{2.1}^*-t_1\right)}-1\right)U_2}{\theta}+\frac{(s+\delta l)}{\delta^2}\left(\delta t_{3.1}^*-\ln\left(1+\delta t_{3.1}^*\right)\right)+\frac{A}{D(p)}-\frac{cE\left[\tilde{i}_e\right]M^2}{2}\right]=0$$

Since,

$$1-\frac{\delta t_{3.1}^*-\ln\left(1+\delta t_{3.1}^*\right)}{\delta\left(t_{2.1}^*+t_{3.1}^*\right)}+\frac{E\left[\tilde{i}_e\right]M^2}{2\left(t_{2.1}^*+t_{3.1}^*\right)}>1-\frac{\delta t_{3.1}^*-\ln\left(1+\delta t_{3.1}^*\right)}{\delta t_{3.1}^*}+\frac{E\left[\tilde{i}_e\right]M^2}{2\left(t_{2.1}^*+t_{3.1}^*\right)}=\frac{\ln\left(1+\delta t_{3.1}^*\right)}{\delta t_{3.1}^*}+\frac{E\left[\tilde{i}_e\right]M^2}{2\left(t_{2.1}^*+t_{3.1}^*\right)}>0$$

It follows from (20) that $\partial E\left[\Pi_1\left(p, t_{2.1}^*, t_{3.1}^*\right)\right] / \partial p = 0$ has a solution if

$D(p)+D'(p)(p-c)<0$ (cf. Dye; 2007)

Furthermore, following Dye (2007) we have,

$$\frac{\partial^2 E\left[\Pi_1\left(p,t_{2.1}^*,t_{3.1}^*\right)\right]}{\partial p^2} = \left[2D'(p)+pD''(p)\right]\left[1-\frac{\delta t_{3.1}^*-\ln\left(1+\delta t_{3.1}^*\right)}{\delta\left(t_{2.1}^*+t_{3.1}^*\right)}+\frac{E\left[\tilde{i}_e\right]M^2}{2\left(t_{2.1}^*+t_{3.1}^*\right)}\right]$$

$$-\frac{D''(p)}{\left(t_{2.1}^*+t_{3.1}^*\right)}\left[c\left(t_{2.1}^*+t_{3.1}^*\right)\left[1-\frac{\delta t_{3.1}^*-\ln\left(1+\delta t_{3.1}^*\right)}{\delta\left(t_{2.1}^*+t_{3.1}^*\right)}\right]\right.$$

$$+\frac{\left(e^{\theta\left(t_{2.1}^*-t_1\right)}-\theta\left(t_{2.1}^*-t_1\right)-1\right)U_1}{\theta^2}+\frac{E\left[\tilde{h}\right]t_1^2}{2}+\frac{cE\left[\tilde{i}_p\right]\left(t_1-M\right)^2}{2} \tag{21}$$

$$\left.+\frac{\left(e^{\theta\left(t_{2.1}^*-t_1\right)}-1\right)U_2}{\theta}+\frac{(s+\delta l)}{\delta^2}\left(\delta t_{3.1}^*-\ln\left(1+\delta t_{3.1}^*\right)\right)+\frac{A}{D(p)}-\frac{cE\left[\tilde{i}_e\right]M^2}{2}\right]<0$$

Thus, there exist unique optimal selling price p_1^* that satisfy (10). Note that the lower bound of optimal selling price (say p_l) is the solution of $D(p)+D'(p)(p-c)=0$ such that $\partial E\left[\Pi_1\left(p,t_{2.1}^*,t_{3.1}^*\right)\right]/\partial p=0$.

Case 2: $t_1 < M \le t_2$

From (6), the expected value of the total profit during the replenishment cycle per unit time can be written as follows.

$$E\left[\Pi_2(p,t_2,t_3)\right]=(p-c)D(p)-\frac{D(p)}{(t_2+t_3)}\left[\left\{\frac{\left(e^{\theta(t_2-t_1)}-\theta(t_2-t_1)-1\right)}{\theta^2}\left(E\left[\tilde{h}\right]+c\theta\right)\right.\right.$$
$$+\frac{E\left[\tilde{h}\right]t_1^2}{2}+\frac{E\left[\tilde{h}\right]t_1}{\theta}\left(e^{\theta(t_2-t_1)}-1\right)+\frac{\left(s+\delta(p-c+l)\right)}{\delta^2}\left(\delta t_3-\ln(1+\delta t_3)\right)$$
$$\left.\left.+\frac{\left(e^{\theta(t_2-M)}-\theta(t_2-M)-1\right)}{\theta^2}cE\left[\tilde{i}_p\right]+\frac{A}{D(p)}\right\}-\frac{pE\left[\tilde{i}_p\right]M^2}{2}\right] \tag{22}$$

For known p, the necessary conditions for the expected total profit function in (22) to be maximum are $\partial E\left[\Pi_2(p,t_2,t_3)\right]/\partial t_2=0$ and $\partial E\left[\Pi_2(p,t_2,t_3)\right]/\partial t_3=0$, which give

$$\frac{\partial E\left[\Pi_2(p,t_2,t_3)\right]}{\partial t_2}=\frac{1}{(t_2+t_3)}\left[\Pi_2^{(1)}(p,t_2,t_3)-D(p)\left\{U_3\left(e^{\theta(t_2-t_1)}-1\right)+U_4e^{\theta(t_2-t_1)}+U_5\left(e^{\theta(t_2-M)}-1\right)\right\}\right]=0 \tag{23}$$

$$\frac{\partial E\left[\Pi_2(p,t_2,t_3)\right]}{\partial t_3}=\frac{1}{(t_2+t_3)}\left[\Pi_2^{(1)}(p,t_2,t_3)-D(p)\left\{\frac{V(p)t_3}{1+\delta t_3}\right\}\right]=0 \tag{24}$$

where $U_3=\dfrac{E\left[\tilde{h}\right]+c\theta}{\theta}, U_4=E\left[\tilde{h}\right]t_1, U_5=\dfrac{cE\left[\tilde{i}_p\right]}{\theta}$, $V(p)=s+\delta(p-c+l)$

and $\Pi_2^{(1)}(p,t_2,t_3)=(p-c)D(p)-E\left[\Pi_2(p,t_2,t_3)\right]$

We want to find the pair of values (t_2,t_3) which satisfies Eqs. (23) and (24) simultaneously. From Eqs. (23) and (24) we obtain respectively,

$$\Pi_2^{(1)}(p,t_2,t_3)=D(p)\left\{U_3\left(e^{\theta(t_2-t_1)}-1\right)+U_4e^{\theta(t_2-t_1)}+U_5\left(e^{\theta(t_2-M)}-1\right)\right\} \tag{25}$$

$$\Pi_2^{(1)}(p,t_2,t_3)=D(p)\left\{\frac{V(p)t_3}{1+\delta t_3}\right\} \tag{26}$$

Equating right hand side of Eq. (25) and Eq. (26) we have

$$U_3\left(e^{\theta(t_2-t_1)}-1\right)+U_4e^{\theta(t_2-t_1)}+U_5\left(e^{\theta(t_2-M)}-1\right)=\frac{V(p)t_3}{1+\delta t_3} \tag{27}$$

For convenience, let $K_2(t_2)$ denote the left hand side of Eq. (27), that is,

$$K_2(t_2)=U_3\left(e^{\theta(t_2-t_1)}-1\right)+U_4e^{\theta(t_2-t_1)}+U_5\left(e^{\theta(t_2-M)}-1\right)$$

which implies

$$t_3=\frac{K_2(t_2)}{V(p)-\delta K_2(t_2)} \tag{28}$$

Thus, t_3 is a function of t_2 and p.

Now, we substitute $\Pi_2^{(1)}(p,t_2,t_3)$ into Eq. (25) and after little calculation we obtain

$$\frac{D(p)}{(t_2+t_3)}\left\{\frac{U_3\left(e^{\theta(t_2-t_1)}-1\right)\left(1-\theta(t_2+t_3)\right)}{\theta}-U_3(t_2-t_1)+\frac{E\left[\tilde{h}\right]t_1^2}{2}+\frac{U_4e^{\theta(t_2-t_1)}\left(1-\theta(t_2+t_3)\right)}{\theta}\right. \tag{29}$$
$$\left.-\frac{U_4}{\theta}+\frac{V(p)\left(\delta t_3-\ln(1+\delta t_3)\right)}{\delta^2}+\frac{U_5\left(e^{\theta(t_2-M)}-1\right)\left(1-\theta(t_2+t_3)\right)}{\theta}-U_5(t_2-M)+\frac{A}{D(p)}-\frac{pE\left[\tilde{i}_e\right]M^2}{2}\right\}=$$

Motivated by Eq. (29), we assume an auxiliary function, say $F_2(t_2), t_2\in[M,\infty)$, where

$$F_2(t_2)=\frac{U_3\left(e^{\theta(t_2-t_1)}-1\right)\left(1-\theta(t_2+t_3)\right)}{\theta}-U_3(t_2-t_1)+\frac{E\left[\tilde{h}\right]t_1^2}{2}+\frac{U_4e^{\theta(t_2-t_1)}\left(1-\theta(t_2+t_3)\right)}{\theta}$$
$$-\frac{U_4}{\theta}+\frac{V(p)\left(\delta t_3-\ln(1+\delta t_3)\right)}{\delta^2}+\frac{U_5\left(e^{\theta(t_2-M)}-1\right)\left(1-\theta(t_2+t_3)\right)}{\theta}-U_5(t_2-M)+\frac{A}{D(p)}-\frac{pE\left[\tilde{i}_e\right]M^2}{2}.$$

and t_3 is given as in Eq. (28). Differentiating $F_2(t_2)$ with respect to t_2, and using the relation in Eqs. (27) and (28) we get

$$\frac{dF_2(t_2)}{dt_2}=-\left[(U_3+U_4)e^{\theta(t_2-t_1)}+e^{\theta(t_2-M)}U_5\right]\theta(t_2+t_3)<0 \tag{30}$$

Thus, $F_2(t_2)$ is strictly decreasing function with respect to $t_2\in[M,\infty)$ and it can be shown that as t_2 gets larger $F_2(t_2)$ approaches to $-\infty$.

Summarizing the above arguments and as discussed earlier in case 1, we can obtain the following result.

Lemma 4.2: For known p, we have

(a) If $F_2(M)\geq 0$ then there exist unique pair of values $(t_2,t_3)=(t_{2.2},t_{3.2})$ which satisfies (23) and (24).

(b) If $F_2(M)<0$ then the optimal value occurs at point $(t_2,t_3)=\left(M,\dfrac{K_2(M)}{V(p)-\delta K_2(M)}\right)$.

Suppose $\left(t_{2.2}^*, t_{3.2}^*\right)$ denotes the optimal value of $\left(t_2, t_3\right)$ for case 2 then we can obtain following result.

Theorem 4.2: For known p, the expected value of total profit function $E\left[\Pi_2\left(p, t_2, t_3\right)\right]$ is concave and attains its global maximum at point $\left(t_2, t_3\right) = \left(t_{2.2}^*, t_{3.2}^*\right)$.

Proof: Analogous to theorem 4.1.

Next, the condition for existence and uniqueness for the optimal selling price can be derived analogously as in Case 1. Consequently, there exist unique optimal selling price, denoted by p_2^*, that satisfy $\partial E\left[\Pi_1\left(p, t_{2.2}^*, t_{3.2}^*\right)\right] / \partial p = 0$ which is given by

$$\frac{\partial E\left[\Pi_1\left(p, t_{2.2}^*, t_{3.2}^*\right)\right]}{\partial p} = \left[D(p) + D'(p)(p-c)\right]\left[1 - \frac{\delta t_{3.2}^* - \ln\left(1+\delta t_{3.2}^*\right)}{\delta\left(t_{2.2}^* + t_{3.2}^*\right)} + \frac{E\left[\tilde{i}_e\right] M^2}{2\left(t_{2.2}^* + t_{3.2}^*\right)}\right]$$
$$-\frac{D'(p)}{\left(t_{2.2}^* + t_{3.2}^*\right)}\left[\frac{\left(e^{\theta\left(t_{2.2}^* - t_1\right)} - \theta\left(t_{2.2}^* - t_1\right) - 1\right) U_3}{\theta} + \frac{E\left[\tilde{h}\right] t_1^2}{2} + \frac{\left(e^{\theta\left(t_{2.2}^* - t_1\right)} - 1\right)}{\theta} U_4 + \frac{(s + \delta l)}{\delta^2}\left(\delta t_{3.2}^* - \ln\left(1 + \delta t_{3.2}^*\right)\right)\right.$$
$$\left. + \frac{\left(e^{\theta\left(t_{2.2}^* - M\right)} - \theta\left(t_{2.2}^* - M\right) - 1\right)}{\theta} U_5 + \frac{A}{D(p)} - \frac{cE\left[\tilde{i}_e\right] M^2}{2}\right] \tag{31}$$

Case 3: $M > t_2$

From Eq. (6), the expected value of the total profit during the replenishment cycle per unit time can be written as follows.

$$E\left[\Pi_3\left(p, t_2, t_3\right)\right] = (p-c) D(p) - \frac{D(p)}{\left(t_2 + t_3\right)}\left[\left\{\frac{\left(e^{\theta\left(t_2 - t_1\right)} - \theta\left(t_2 - t_1\right) - 1\right)}{\theta^2}\left(E\left[\tilde{h}\right] + c\theta\right)\right.\right.$$
$$+ \frac{E\left[\tilde{h}\right] t_1^2}{2} + \frac{E\left[\tilde{h}\right] t_1}{\theta}\left(e^{\theta\left(t_2 - t_1\right)} - 1\right) + \frac{\left(s + \delta\left(p - c + l\right)\right)}{\delta^2}\left(\delta t_3 - \ln\left(1 + \delta t_3\right)\right)$$
$$\left.\left. + \frac{A}{D(p)}\right\} - pE\left[\tilde{i}_e\right] t_2\left(M - t_2/2\right)\right] \tag{32}$$

For known p, the necessary conditions for the expected total profit function in (32) to be maximum are $\partial E\left[\Pi_3\left(p, t_2, t_3\right)\right] / \partial t_2 = 0$ and $\partial E\left[\Pi_3\left(p, t_2, t_3\right)\right] / \partial t_3 = 0$, which give

$$\frac{\partial E\left[\Pi_3\left(p, t_2, t_3\right)\right]}{\partial t_2} = \frac{1}{\left(t_2 + t_3\right)}\left[\Pi_3^{(1)}\left(p, t_2, t_3\right) - D(p)\left\{U_3\left(e^{\theta\left(t_2 - t_1\right)} - 1\right) + U_4 e^{\theta\left(t_2 - t_1\right)} - pE\left[\tilde{i}_e\right]\left(M - t_2\right)\right\}\right] = 0 \tag{33}$$

$$\frac{\partial E\left[\Pi_3\left(p, t_2, t_3\right)\right]}{\partial t_3} = \frac{1}{\left(t_2 + t_3\right)}\left[\Pi_3^{(1)}\left(p, t_2, t_3\right) - D(p)\left\{\frac{V(p) t_3}{1 + \delta t_3}\right\}\right] = 0 \tag{34}$$

where $\Pi_3^{(1)}(p,t_2,t_3) = (p-c)D(p) - E\left[\Pi_3(p,t_2,t_3)\right]$

Proceed in the same manner as Case 2 it follows from Eqs. (33) and (34),

$$t_3 = \frac{K_3(t_2)}{V(p) - \delta K_3(t_2)} \tag{35}$$

where $K_3(t_2) = U_3\left(e^{\theta(t_2-t_1)} - 1\right) + U_4 e^{\theta(t_2-t_1)} - pE\left[\tilde{i}_e\right](M - t_2)$

Now, we substitute $\Pi_3^{(1)}(p,t_2,t_3)$ into Eq. (33) and after little calculation we obtain

$$\begin{aligned}\frac{D(p)}{(t_2+t_3)}\Bigg\{&\frac{U_3\left(e^{\theta(t_2-t_1)}-1\right)\left(1-\theta(t_2+t_3)\right)}{\theta} - U_3(t_2-t_1) + \frac{E\left[\tilde{h}\right]t_1^2}{2} + \frac{U_4 e^{\theta(t_2-t_1)}\left(1-\theta(t_2+t_3)\right)}{\theta}\\ &-\frac{U_4}{\theta} + \frac{V(p)\left(\delta t_3 - \ln(1+\delta t_3)\right)}{\delta^2} + \frac{A}{D(p)} - pE\left[\tilde{i}_e\right]\left[t_2(M - t_2/2) - (M-t_2)(t_2+t_3)\right]\Bigg\} = 0\end{aligned} \tag{36}$$

Motivated by Eq. (36), we assume an auxiliary function, say $F_3(t_2), t_2 \in [t_1, M]$, where

$$\begin{aligned}F_3(t_2) = &\frac{U_3\left(e^{\theta(t_2-t_1)}-1\right)\left(1-\theta(t_2+t_3)\right)}{\theta} - U_3(t_2-t_1) + \frac{E\left[\tilde{h}\right]t_1^2}{2} + \frac{U_4 e^{\theta(t_2-t_1)}\left(1-\theta(t_2+t_3)\right)}{\theta}\\ &-\frac{U_4}{\theta} + \frac{V(p)\left(\delta t_3 - \ln(1+\delta t_3)\right)}{\delta^2} - pE\left[\tilde{i}_e\right]\left[t_2(M - t_2/2) - (M-t_2)(t_2+t_3)\right] + \frac{A}{D(p)}\end{aligned} \tag{37}$$

and t_3 is given as in Eq. (35). Differentiating $F_3(t_2)$ with respect to t_2, we get

$$\frac{dF_2(t_2)}{dt_2} = -\left[(U_3+U_4)\theta e^{\theta(t_2-t_1)} + pE\left[\tilde{i}_e\right]\right](t_2+t_3) < 0 \tag{38}$$

Thus, $F_3(t_2)$ is strictly decreasing function with respect to $t_2 \in [t_1, M]$.

For notational convenience, let $\Delta_1 = F_3(M)$ and $\Delta_2 = F_3(t_1)$

Lemma 4.3: For known p, we have

(a) If $\Delta_1 \le 0 \le \Delta_2$ then there exist unique pair of values $(t_2, t_3) = (t_{2.3}, t_{3.3})$ which satisfies Eqs. (33-34).

(b) If $\Delta_2 < 0$ then the optimal value occurs at point $(t_2,t_3) = \left(t_1, \dfrac{K_3(t_1)}{V(p) - \delta K_3(t_1)}\right)$.

(c) If $\Delta_1 > 0$ then the optimal value occurs at point $(t_2,t_3) = \left(M, \dfrac{K_3(M)}{V(p) - \delta K_3(M)}\right)$.

Proof: (a) Since $F_3(t_2)$ is strictly decreasing function with respect to $t_2 \in [t_1, M]$ and by assumption $\Delta_1 \le 0 \le \Delta_2 \Rightarrow F_3(M) \le 0 \le F_3(t_1)$ we can find t_2 (say $t_{2.3} \in [t_1, M]$) such that $F_3(t_{2.3}) = 0$. The corresponding value of $t_3 = t_{3.3}$ can be obtained from Eq. (34). This implies that the pair of values (t_2, t_3) which satisfies (33) and (34) not only exists but also is unique.

(b) If $\Delta_2 < 0$ then $F_3(t_1) < 0$. Since $F_3(t_2)$ is strictly decreasing function of $t_2 \in [t_1, M]$; $\partial E[\Pi_3(p, t_2, t_3)]/\partial t_2 < 0$, $\forall t_2 \in [t_1, M]$. Hence, optimal value occurs at point $t_2 = t_1$ and corresponding optimal value of t_3 can be found from Eq. (35) and is given by $\dfrac{K_3(t_1)}{V(p) - \delta K_3(t_1)}$.

(c) If $\Delta_1 > 0$ then $F_3(M) > 0$. Since $F_3(t_2)$ is strictly decreasing function of $t_2 \in [t_1, M]$; $\partial E[\Pi_3(p, t_2, t_3)]/\partial t_2 = D(p)/(t_2 + t_3) F_3(t_2) > 0$, $\forall t_2 \in [t_1, M]$. Thus $E[\Pi_3(p, t_2, t_3)]$ is strictly increasing over $[t_1, M]$. Hence, maximum value occurs at point $t_2 = M$ and corresponding optimal value of t_3 can be found from Eq. (35) and is given by $\dfrac{K_3(M)}{V(p) - \delta K_3(M)}$.

Suppose $(t_{2.3}^*, t_{3.3}^*)$ denotes the optimal value of (t_2, t_3) for Case 3 then we can obtain following result.

Theorem 4.3: For known *p*, the expected value of total profit function $E[\Pi_3(p, t_2, t_3)]$ is concave and attains its global maximum at point $(t_2, t_3) = (t_{2.3}^*, t_{3.3}^*)$.

Proof: Similar to theorem 4.1.

Next, the condition for existence and uniqueness for the optimal selling price can be obtained similar manner as in Case 1. Therefore, there exist unique optimal selling price, denoted by p_3^*, that satisfy $\partial E[\Pi_1(p, t_{2.3}^*, t_{3.3}^*)]/\partial p = 0$ where $\partial E[\Pi_1(p, t_{2.3}^*, t_{3.3}^*)]/\partial p$ given as,

$$
\begin{aligned}
\frac{\partial E\left[\Pi_1\left(p, t_{2.3}^*, t_{3.3}^*\right)\right]}{\partial p} &= \left[D(p) + D'(p)(p-c)\right] \\
&\times \left[1 - \frac{\delta t_{3.3}^* - \ln\left(1+\delta t_{3.3}^*\right)}{\delta\left(t_{2.3}^* + t_{3.3}^*\right)} + \frac{E\left[\tilde{i}_e\right] t_{2.3}^*\left(M - t_{2.3}^*/2\right)}{2\left(t_{2.3}^* + t_{3.3}^*\right)}\right] \\
&- \frac{D'(p)}{\left(t_{2.3}^* + t_{3.3}^*\right)} \left[\frac{\left(e^{\theta\left(t_{2.3}^* - t_1\right)} - \theta\left(t_{2.3}^* - t_1\right) - 1\right) U_3}{\theta} + \frac{E\left[\tilde{h}\right] t_1^2}{2}\right.
\end{aligned} \tag{39}
$$

$$
\left. + \frac{\left(e^{\theta\left(t_{2.3}^* - t_1\right)} - 1\right) U_4}{\theta} + \frac{(s + \delta l)}{\delta^2}\left(\delta t_{3.3}^* - \ln\left(1 + \delta t_{3.3}^*\right)\right) + \frac{A}{D(p)} - \frac{cE\left[\tilde{i}_e\right] E\left[\tilde{i}_e\right] t_{2.3}^*\left(M - t_{2.3}^*/2\right)}{2}\right]
$$

Based on the concavity behavior of objective function with respect to the decision variables the following algorithmic procedure was developed to identify global optimal solutions for (p, t_2, t_3).

Algorithm 4.1:

Step 1: Input the values of all parameters. Select membership functions for holding cost rate, interest paid rate and interest earned rate with appropriate parametric values.

Step 2: Set $k = 1$ and initialize the value of $p^{(k)} = p_l$, which is a solution of $D(p) + (p-c)D'(p) = 0$.

Step 3: Compare the values of M and t_1. If $M \le t_1$, then go to Step 4 otherwise go to Step 5.

Step 4: Calculate $\Delta = F_1(t_1)$ by Eq. (16). Execute any one of the following cases (4.1), (4.2).

(4.1) If $\Delta \ge 0$, obtain the values of $\left(t_2^{(k)}, t_3^{(k)}\right)$ by solving Eqs. (8) and (9). Substitute the values of $\left(t_2^{(k)}, t_3^{(k)}\right)$ into Eq. (10) and solve to obtain the value of $p_1^{(k)}$. Set $p^{(k+1)} = p_1^{(k)}$.

(4.2) If $\Delta < 0$, then set $\left(t_2^{(k)}, t_3^{(k)}\right) = \left(t_1, \dfrac{U_2}{\theta V(p) - U_2}\right)$. Substitute the values of $\left(t_2^{(k)}, t_3^{(k)}\right)$ into Eq. (10) and solve to obtain the value of $p_1^{(k)}$. Set $p^{(k+1)} = p_1^{(k)}$.

Step 5: Calculate $\Delta_1 = F_3(M)$ and $\Delta_2 = F_3(t_1)$ by Eq. (37). Execute any one of the following cases (5.1), (5.2), (5.3).

(5.1) If $\Delta_1 \le 0 \le \Delta_2$, obtain the values of $\left(t_2^{(k)}, t_3^{(k)}\right)$ by solving Eqs. (33) and (34). Substitute the values of $\left(t_2^{(k)}, t_3^{(k)}\right)$ into Eq. (39) and solve to obtain the value of $p_1^{(k)}$. Set $p^{(k+1)} = p_1^{(k)}$.

(5.2) If $\Delta_2 < 0$ then set $\left(t_2^{(k)}, t_3^{(k)}\right) = \left(t_1, \dfrac{K_3(t_1)}{V(p) - \delta K_3(t_1)}\right)$. Substitute the values of $\left(t_2^{(k)}, t_3^{(k)}\right)$ into Eq. (39) and solve to obtain the value of $p_1^{(k)}$. Set $p^{(k+1)} = p_1^{(k)}$.

(5.3) If $\Delta_1 > 0$ then perform following steps.

(5.3.1) Obtain the values of $\left(t_{2.2}^{(k)}, t_{3.2}^{(k)}\right)$ by solving Eqs. (25) and (26). Calculate $E\left[\Pi_2\left(p^{(k)}, t_{2.2}^{(k)}, t_{3.2}^{(k)}\right)\right]$.

(5.3.2) Set $\left(t_2^{(k)}, t_3^{(k)}\right) = \left(M, \dfrac{K_3(M)}{V(p) - \delta K_3(M)}\right)$ and then calculate $E\left[\Pi_3\left(p^{(k)}, t_2^{(k)}, t_3^{(k)}\right)\right]$.

(5.3.3) If $E\left[\Pi_2\left(p^{(k)}, t_{2.3}^{(k)}, t_{3.3}^{(k)}\right)\right] \ge E\left[\Pi_3\left(p^{(k)}, t_2^{(k)}, t_3^{(k)}\right)\right]$ then set $\left(t_2^{(k)}, t_3^{(k)}\right) = \left(t_{2.2}^{(k)}, t_{3.2}^{(k)}\right)$

otherwise $\left(t_2^{(k)}, t_3^{(k)}\right) = \left(M, \dfrac{K_3(M)}{V(p)-\delta K_3(M)}\right)$.

(5.3.4) Substitute the values of $\left(t_2^{(k)}, t_3^{(k)}\right)$ into Eq. (31) (or Eq. (39)) and solve to obtain the value of $p_1^{(k)}$. Set $p^{(k+1)} = p_1^{(k)}$.

Step 6: If $\left|p^{(k+1)} - p^{(k)}\right| \le \text{tolerance}$, then set $\left(p_i^*, t_{2.i}^*, t_{3.i}^*\right) = \left(p^{(k+1)}, t_2^{(k)}, t_3^{(k)}\right)$, i = 1 or 2 or 3, and go to Step 7. Otherwise set $k = k + 1$ and go to Step 2.

Step 7: Compute $E\left[\Pi\left(p^*, t_2^*, t_3^*\right)\right] = \max_{1\le i\le 3} E\left[\Pi_i\left(p_i^*, t_{2.i}^*, t_{3.i}^*\right)\right]$ and corresponding T^* and Q^*.

5. Numerical Example

To illustrate the above solution methodology on fuzzy EVM developed in this paper, four examples (mostly of the data from Geetha and Uthayakumar; 2010) are considered. In this paper, we have assumed the holding cost rate, interest paid rate and interest earned rate as fuzzy variable. Without loss of generality, we characterize holding cost rate and interest paid rate as Trapezoidal and Triangle fuzzy variable which are given by respectively $\tilde{h} = (12,14,16,18)$ and $\tilde{i_p} = (0.13, 0.15, 0.17)$. Furthermore, interest earned rate $\left(\tilde{i_e}\right)$ is characterized by arbitrary fuzzy variable ξ with $Cr\{\xi \ge t\}$ given by

$$Cr\{\xi \ge t\} = \begin{cases} 0 & \text{for } t > 0.13 \\ 5\sqrt{t-0.12} & \text{for } 0.12 \le t \le 0.13 \\ 1-1250(0.12-t)^2 & \text{for } 0.1 \le t \le 0.12 \\ 1 & \text{otherwise} \end{cases}$$

Example 1: Consider an inventory system with following parametric values in appropriate units. A = 250, c = 80, θ = 0.08, δ = 0.56, t_1 = 0.0685, M = 0.1233, c_s = 30, c_l = 25, D (p) = 2000 – 2.8 p, $E\left[\tilde{h}\right] = 15$, $E\left[\tilde{i_p}\right] = 0.15$, $E\left[\tilde{i_e}\right] = 0.12$. By solving $D(p) + (p-c)D'(p)$ = 0, we obtain $p_l = p^{(1)}$ = 397.14. Executing the procedure proposed in computational algorithm 4.1 we find that $\left(p^*, t_2^*, t_3^*\right) = (397.14, 0.09264, 0.00039)$ and hence corresponding $E\left[\Pi\left(p^*, t_2^*, t_3^*\right)\right]$ = \$ 281,547.03, T^* = 0.09303 and Q^* = 82.60.

Example 2: The same set of input data are considered as in the Example 1 except that t_1 = 0.0904, M = 0.1096. Applying the procedure proposed in computational algorithm 4.1 we find that $\left(p^*, t_2^*, t_3^*\right)$ = $(397.20, 0.09422, 0.00319)$ and hence corresponding $E\left[\Pi\left(p^*, t_2^*, t_3^*\right)\right]$ = \$ 280,996.68, T^* = 0.09741 and Q^* = 83.92.

Example 3: The identical set of input data are considered as in the Example 1 except t_1 = 0.5014, M = 0.0548. Performing the procedure proposed in computational algorithm 4.1 we find that $\left(p^*, t_2^*, t_3^*\right)$ = $(400.05, 0.5014, 0.02907)$ and hence corresponding $E\left[\Pi\left(p^*, t_2^*, t_3^*\right)\right]$ = \$ 275,980.40, T^* = 0.53047 and Q^* = 466.61.

Example 4: The data are same as in the Example 1 except that $t_1 = 0.0822$, $M = 0.0548$. Implementing the procedure proposed in computational algorithm 4.1 we find that $\left(p^*, t_2^*, t_3^*\right) = \left(397.61, 0.11788, 0.01253\right)$ and hence corresponding $E\left[\Pi\left(p^*, t_2^*, t_3^*\right)\right] =$ \$ 279,175.15, $T^* = 0.13041$ and $Q^* = 115.64$.

To show the efficiency of proposed computational algorithm 4.1, we run the algorithm with starting value of $p = 360$. The graph (Fig. 1) shows clear concave function of t_2 and t_3 for given p. Consequently, the obtained solution is a global maximum solution.

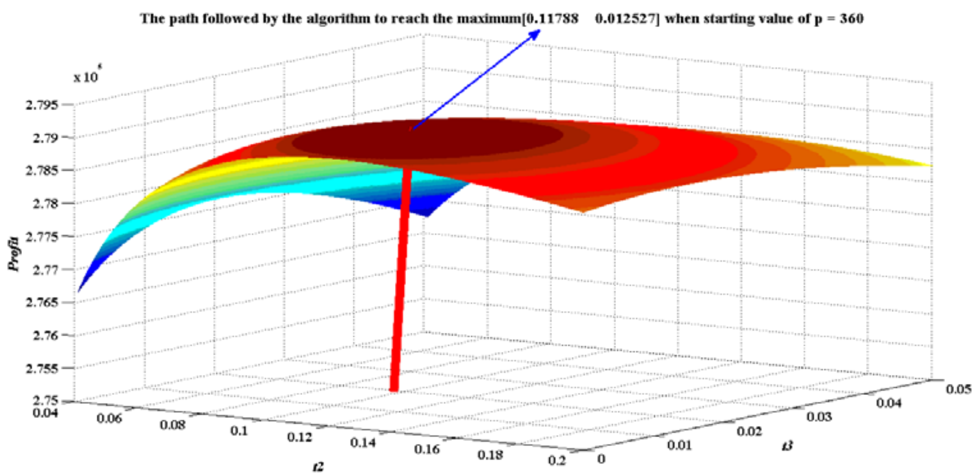

Fig. 1. Profit function (Example 4) with respect to t_2 and t_3

6. Conclusion

According to the model of Geetha and Uthayakumar (2010), a new fuzzy EVM with generalized price sensitive demand is formulated. In contrast to previous studies, we characterized the holding cost rate, interest paid rate and interest earned rate as independent fuzzy variables to tackle the reality in more effective way. A solution methodology along with some useful theoretical results followed by an efficient computational algorithm is developed to determine the optimal pricing and inventory decisions. The extended model is more effective as it can help the decision maker in subjective decisions with control on selling price. In future research on this problem, it would be interesting to consider other parameters viz. variable demand rate, partial backlogging rate etc. as fuzzy or fuzzy stochastic.

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Appendix. Fuzzy Preliminary

Let X be a nonempty set, P (X) the power set of X and Cr be a credibility measure. Then the triplet (X, P(X), Cr) is called a credibility space. Following Liu [9], the fuzzy variable is defined as follows

Definition 2.1: A fuzzy variable is defined as a function from a credibility space (X, P(X), Cr) to the set of real numbers.

Let ξ be a fuzzy variable defined on the credibility space (X, P(X), Cr). Then its membership function μ is derived from the credibility measure through

μ(x) = (2Cr {ξ = x}) $\wedge$ 1, x $\in$ R.

Definition 2.2: Let ξ be a fuzzy variable with membership function μ. Then for any Borel set B of real numbers,

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^C}\mu(x)\right) \tag{A.1}$$

Definition 2.7 (Liu and Liu [10])**:** Let ξ and *r* be a fuzzy variable and crisp number respectively then the expected value $E[\xi]$ is defined as

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r\}\,dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}\,dr \tag{A.2}$$

provided that at least one of the two integral is finite. Especially, if ξ is a positive fuzzy variable then

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r\}\,dr \tag{A.3}$$

Definition 2.4: An *trapezoidal fuzzy variable* ξ is specified by four parameters (a_1, a_2, a_3, a_4) where $a_1 < a_2 < a_3 < a_4$ and is characterized by the membership function μ_ξ, given by

$$\mu_\xi(x) = \begin{cases} \left(\dfrac{x-a_1}{a_2-a_1}\right) & \text{for } a_1 \leq x \leq a_2 \\ 1 & \text{for } a_2 \leq x \leq a_3 \\ \left(\dfrac{a_4-x}{a_4-a_3}\right) & \text{for } a_3 \leq x \leq a_4 \\ 0 & \text{otherwise} \end{cases} \tag{A.4}$$

If ξ be a trapezoidal fuzzy variable and *t* be any crisp number. Then $Cr\{\xi \geq t\}$ is given by

$$Cr\{\xi \geq t\} = \begin{cases} 0 & \text{for } t < a_1 \\ 1-0.5\left(\dfrac{t-a_1}{a_2-a_1}\right) & \text{for } a_1 \leq t \leq a_2 \\ 0.5 & \text{for } a_2 \leq t \leq a_3 \\ 0.5\left(\dfrac{a_4-t}{a_4-a_3}\right) & \text{for } a_3 \leq t \leq a_4 \\ 1 & \text{otherwise} \end{cases} \tag{A.5}$$

Making use of (A.3) we determine the expected value of the trapezoidal fuzzy variable $\xi = (a_1, a_2, a_3, a_4)$ to be,

$$E[\xi] = \frac{a_1 + a_2 + a_3 + a_4}{4} \tag{A.6}$$

Note: If we take $a_2 = a_3$ in trapezoidal fuzzy variable then it reduced to triangle fuzzy variable.

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