

An integrated multi-stage supply chain inventory model with imperfect production process

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CHRONICLE

Article history:

Received October 14 2014
Received in Revised Format
February 10 2015
Accepted March 29 2015
Available online
April 5 2015

Keywords:

Integrated supply chain
Multi-buyer
Rework
Shipment

ABSTRACT

This paper deals with an integrated multi-stage supply chain inventory model with the objective of cost minimization by synchronizing the replenishment decisions for procurement, production and delivery activities. The supply chain structure examined here consists of a single manufacturer with multi-buyer where manufacturer orders a fixed quantity of raw material from outside suppliers, processes the materials and delivers the finished products in unequal shipments to each customer. In this paper, we consider an imperfect production system, which produces defective items randomly and assumes that all defective items could be reworked. A simple algorithm is developed to obtain an optimal production policy, which minimizes the expected average total cost of the integrated production-inventory system.

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1. Introduction

The globalization of world economy and increasing competitive markets have compelled business to improve the performance of the supply chain that can promptly respond to customer requirements and make sure the availability of the products and worldwide services to the customer. Shipment of the product in small lots decreases the inventory holding cost but raises set-up, ordering and transportation costs. Conversely, shipment in larger lots increases inventory holding cost but reduces the other costs, and scheduling interference results due to limited storage space for both the manufacturer and the buyers. Coordination of the scheduling of these stages is essential to take competitive advantages as it reduces overall supply chain cost.

A large numbers of research works have been concentrated on the buyer-vendor integrated inventory model. Goyal (1977) developed a joint economic lot-size model for single buyer and single vendor with infinite production rate. Later, Banerjee (1986) generalized the model by considering finite rate of production for the product with “lot for lot” shipment policy. The research related to integrated vendor – buyer (IVB) models prior to 1989 is well reviewed in the paper of Goyal and Gupta (1989). Afterwards, Lu (1995) suggested an optimal policy in which delivery quantity to the customer is identical at each shipment. Goyal (1995) relaxed the restriction of identical shipments and considered different shipments

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policy in which successive shipments within a production batch are increasing by a constant factor. Later Hill (1997) extended this shipment policy more generally by allowing the geometric growth factor as a decision variable. Hill (1999) and Goyal and Nebebe (2000) kept researching on IVB systems to obtain the best optimal results by considering alternative policies. All the previous studies cover IVB models without considering the raw material procurement.

Some researchers developed integrated procurement–production (IPP) systems to minimize the total cost by determining the raw material procurement lot size and the manufacturing batch size without taking the buyer's ordering quantity or the inventory holding cost into consideration (Golhar & Sarker, 1992; Jamal & Sarker, 1993; Sarker & Parija, 1994). Lee (2005) proposed an integrated inventory model for a single manufacturer, single-buyer supply chain problem by jointly considering IVB and IPP systems.

A new area of integrated supply chain, i.e., single vendor and multi buyer was suggested by Joglekar and Tharthare (1990) and they presented an alternate solution of the same problem proposed by Banerjee (1986). They refined Joint Economic Lot Size (JELS) by breaking set-up cost into vendors' order processing and handling cost per production run setup cost and named this approach as the Individual Responsible and Rational Decision (IRRD). They believed this approach could help the vendor and the buyers take their individual rational decisions. Viswanathan and Piplani (2001) developed a one-vendor multi-buyer supply chain model for a single item to study the advantage of synchronizing the supply chain inventories through common replenishment time periods without considering the inventory of the vendor. Hoque (2008) developed the optimal solution procedures of three models for single-vendor multi-buyer two of which transfer with equal batches and the third with unequal batches of a single product.

Previous studies related to buyer vendor coordination were focused on obtaining minimum total cost by determining raw material procurement lot size, the manufacturing batch size and buyer's ordering quantity, in which the quality-related issues of the product in manufacturing facility are not taken into consideration. However, because of deterioration or other factors, the manufacture process may produce poor quality items. These defective items are either wasted as scraps or sold at a discounted price at the end of the screening process, as many industries having no reworking facility and consequently, the industries lose a big share of profit margin.

Lee et al. (1997) dealt with the imperfect production and quality control issue in a multi-stage production system but they did not employ rework process for defective items. To reduce overall production costs, a production system may have a repair or rework facility. Hayek and Salameh (2001) obtained an optimal operating policy in a lot sizing problem under the effect of reworking of all defective items. Jamal et al. (2004) proposed a single-stage production system in which rework is done under two different operational policies to obtain the optimum batch quantity. In the first policy, the defective items are reworked within the same production cycle. In the second policy, the defective items are accumulated for a certain number of cycles before they are reworked. Giri and Chakraborty (2011) considered a single-vendor single-buyer supply chain model where the production process at the vendor is not perfectly reliable. During a production run, it may shift from an in-control state to an out-of-control state at any random time and produces some defective items. Hsu and Hsu (2012) developed an integrated vendor-buyer inventory model with imperfect product quality and inspection errors. Giri and Sharma (2014) proposed an unequal-sized shipment policy for an integrated production-inventory system under imperfect production process. They assumed that the retailer performs a screening process after getting the ordered quantity and the manufacture incurs a warranty cost.

In this paper, we have developed an integrated supply chain inventory model consists of a single manufacturer and multi-buyer, where manufacturer orders a fixed-quantity of raw material from outside supplier, processes the materials, and delivers in unequal shipments of finished products to each

customer. We also assume that during production process, a portion of defective items is produced randomly which is reworked in each cycle after the end of a production run.

2. Assumption and Notation

To simplify the analysis, we make the following assumptions:

1. Demand and production rates are deterministic and constant.
2. Each buyer estimates individual demand, holding and ordering costs under various cost factors and informs the manufacturer.
3. There is no initial inventory.
4. Shortages are not allowed.
5. All defective items are considered to be repairable and are reworked.
6. No scrap is produced during normal and rework processing.
7. The transport equipment has enough capacity to transport any of the batches to a buyer; and set-up and transportation times are insignificant.

We use the following notations:

D_R	Demand rate of raw material (unit/year)
D	Demand rate of finished goods (unit/year)
D_i	Demand rate of finished goods for i^{th} buyer (unit/yr); $D = \sum_{i=1}^k D_i$
P	Production rate per unit time (units/year)
P_1	Reworking rate per unit time (units/year) ; $P_1 \geq P$
d	Production rate of defective items per unit time (units/year)
x	Portion of defective items produced randomly; $d = Px$
v	Number of production run covered from one procurement of raw material
n	Number of shipments of finished goods
f	Conversion factor of the raw materials to finished goods; $f = D/D_R = vQ_M/Q_R$
Q	Size of the first shipment of finished goods from manufacturer
Q_i	Size of the first shipment of finished goods to i^{th} buyer ($i = 1, 2, \dots, k$)
λ	Proportional increase in size of successive shipments
Q_M	Quantity of finished goods manufactured per set up(units/batch)
S_i	Size of the i^{th} shipment of finished goods from manufacturer
Q_R	Quantity of raw materials required in each batch; $Q_R = vQ_M/f$ (units/order)
A_R	Ordering cost of raw material (\$/order)
A_M	Manufacturing set up cost (\$/batch)
A_{Bi}	i^{th} buyer's ordering cost (\$/order)
C_M	Unit manufacturing cost (\$/unit)
C_o	Raw material cost (\$/unit)
T_1	Production uptime for the proposed EPQ model; $T_1 = Q_M/P$ (in years)
T_2	Time required for reworking of defective items; $T_2 = xQ_M/P_1$ (in years)
C_I	Rework cost (\$/unit)
C_{Ti}	Cost of transporting a batch from the manufacturer to i^{th} buyer
C_R	Unit inventory value of raw material (\$/unit)
C_{MP}	Unit inventory value of manufacturer's finished goods of perfect quality (\$/unit)
C_{MD}	Unit inventory value of defective items (\$/unit)
C_{Bi}	Unit inventory value of i^{th} buyer's incoming inventory (\$/unit) ; $C_{Bi} > C_{MP} > C_{MD} > C_R$
r	Annual capital cost per dollar invested in inventory
TC	Expected total cost per year (in \$)

3. Model Formulation

Here we have considered a manufacturing system, which procures raw materials from suppliers, processes them to convert to finished products. During the production time, x portion of defective items is produced randomly at a rate d .

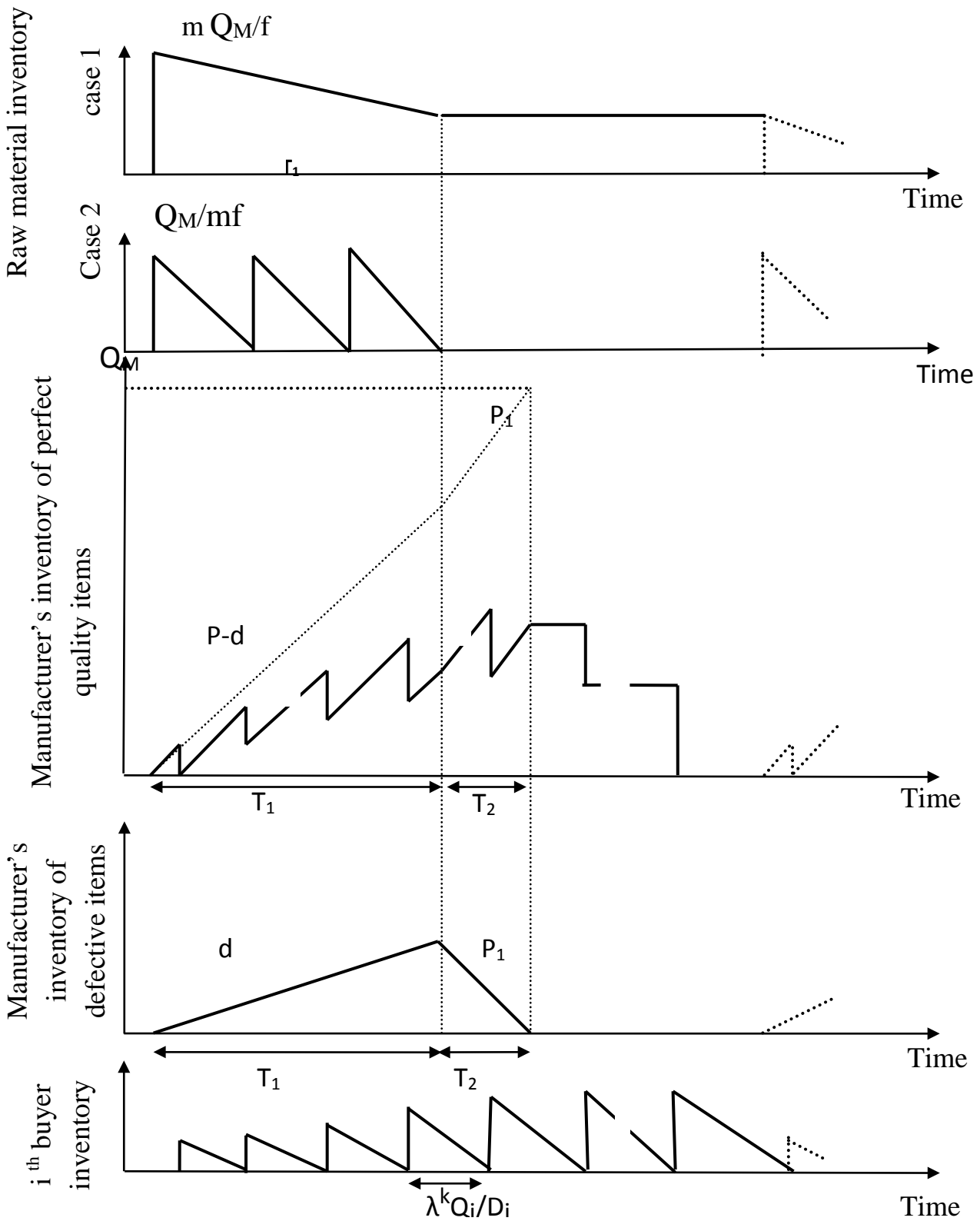


Fig. 1. Inventory of manufacture's raw material, finished items (of perfect and imperfect quality) and buyer's incoming items

All defective items are reworked at a rate P_1 in each cycle at the end of a production run. In order to avoid shortages, we assume that the production rate P has to be larger than the sum of demand rate D and production rate of defective item d . That is: $(P-d-D) > 0$ or $(1-x-D/P) > 0$; where $d = Px$. The demand is met from item of perfect quality. The manufacturer delivers the entire lot Q_M by n unequal shipment of sizes $Q, \lambda Q, \dots, \lambda^{n-1}Q$ to meet the demands of all of the buyers. Since $C_{MP} < C_{Bi}$, the manufacturer delivers a shipment only when the buyers are almost to run out of stock. When the production starts, the total stock in the system is the demand during the time to produce the first shipment and this is minimized when the first shipment is the smallest one. Therefore, we have a sequence of shipment, which increases in size and hence $\lambda \geq 1$. The first shipment from manufacturer to buyers takes place as soon as the required shipment quantity Q is produced, the dispatch of the first shipment will return the manufacturer stock items of perfect quality to zero. The time to produce second shipment, $\lambda Q/P(1-x)$ cannot be greater than the time for the demand process to consume the first shipment, Q/D , and this gives $\lambda \leq P(1-x)/D$. Fig 1 shows the inventory of manufacturer's raw material, manufacturer's finished items of perfect and imperfect quality and buyer's incoming items.

Different types of costs incorporated with manufacturer's and customers' are considered here for the integrated inventory model under an infinite planning horizon.

3.1. Manufacturer's cost

These manufacturer's costs are raw material cost and production cost.

Raw material cost

Manufacturer procures raw materials from the suppliers and converts to finished goods with a conversion factor f . The raw material ordering lot size, Q_R , can be represented as $Q_R = vQ_M/f = v(n+1)Q/f$, where v be the number of production runs covered by a single procurement of raw materials. When $v = 1$ then the raw materials required for each production run is delivered in only one shipment, which is a special case. We consider the two possible ordering situations separately: $v = \{1, 2, \dots, m\}$ for Case 1 and $v = \{1, 1/2, \dots, 1/m\}$ for Case 2, where m is an integer.

Raw material ordering cost per year becomes $A_R \frac{D(\lambda-1)}{vQ(\lambda^n-1)}$

Raw material purchasing cost per year is $C_o \frac{D}{f}$

While evaluating the raw material holding cost, we consider the two possible cases independently. For case 1, each lot size of ordered raw material will meet the demand of m (say) production runs. On the other hand, for case 2 the manufacturer needs to replenish raw materials m times for every production run. The average inventory for each of the cases can be derived as (see Appendix A)

$$\text{Case 1: } Q_{Ravg} = \frac{Q(\lambda^n-1)}{2f(\lambda-1)} \left(\frac{D}{P} + m - 1 \right) \quad (1)$$

$$\text{Case 2: } Q_{Ravg} = \frac{Q(\lambda^n-1)}{2f(\lambda-1)} \left(\frac{D}{mP} \right) \quad (2)$$

Hence the raw material holding cost per year is $rC_R Q_{Ravg}$

Production costs

The manufacturer's production lot size is Q_M ; the lot is delivered by n unequal shipment of sizes $Q, \lambda Q, \dots, \lambda^{n-1}Q$ to meet the demands of all of the buyers. We assume that $Q_i = D_i Q/D$, $\lambda Q_i = D_i \lambda Q/D$, \dots , $\lambda^{n-1} Q_i = D_i \lambda^{n-1} Q/D$ so that $Q = \sum_{i=1}^k Q_i$

Production set up cost per year is $A_M \frac{D(\lambda-1)}{Q(\lambda^n-1)}$

During production uptime the manufacturer's on-hand inventory of perfect quality items are increasing with the rate of $P - d$ and during the reworking period, increasing with the rate of P_1 while depleted by a quantity of $\lambda^{i-1}Q$ for every time interval of $\lambda^{i-1}Q/D$. Therefore, a saw-tooth pattern is built up in the manufacturer's on-hand inventory of perfect quality item during the time interval $[0, T_1 + T_2]$ (see Fig 1). While during the production downtime, the manufacturer's inventory of perfect quality item is flat if no replenishment is taken place and it will be vertically dropped by a quantity of $\lambda^{i-1}Q$ at the end of every shipment to the buyers. Thus, the average inventory of perfect quality can be derived as (see Appendix B)

$$Q_{Pavg} = \frac{Q}{2} \left[\frac{2D}{P(1-x)} + \frac{2\lambda(\lambda^{n-1}-1)}{\lambda^2-1} - \frac{\lambda^{n-1}}{\lambda-1} \left[\frac{D}{P} + D \left(\frac{x}{P} + \frac{x^2}{P_1} \right) \right] \right]. \quad (3)$$

The average inventory of defective item is

$$Q_{Davg} = \frac{DQ(\lambda^n-1)}{2(\lambda-1)} \left(\frac{x}{P} + \frac{x^2}{P_1} \right). \quad (4)$$

Hence the expected holding cost for manufactured items per year is

$$rC_{MP}E[Q_{Pavg}] + rC_{MD}E[Q_{Davg}].$$

Expected reworking cost per year is $C_I DE[x]$ and manufacturing cost per year is $C_M D$.

Hence the expected manufacturer cost per year is

$$TC_M = A_R \frac{D(\lambda-1)}{vQ(\lambda^n-1)} + A_M \frac{D(\lambda-1)}{Q(\lambda^n-1)} + rC_R Q_{Ravg} + rC_{MP}E[Q_{Pavg}] + rC_{MD}E[Q_{Davg}] + C_o \frac{D}{f} + C_I DE[x] + C_M D \quad (5)$$

3.2 Customers' cost

Ordering cost for i^{th} buyer per year is $A_{Bi} \frac{nD(\lambda-1)}{Q(\lambda^n-1)}$

The i^{th} buyer receives batches of sizes $Q_i, \lambda Q_i, \dots, \lambda^{n-1} Q_i$. The average inventory for i^{th} buyer per cycle is $\frac{Q^2 D_i (\lambda^{2n}-1)}{2D^2 (\lambda^2-1)}$.

Hence the inventory holding cost for i^{th} buyer per year is $rC_{Bi} \frac{QD_i(\lambda^n+1)}{2D(\lambda+1)}$

Transportation cost for i^{th} buyer per year is $C_{Ti} \frac{nD(\lambda-1)}{Q(\lambda^n-1)}$

Hence all customers' cost per year is

$$TC_B = \frac{nD(\lambda-1)}{Q(\lambda^n-1)} \sum_{i=1}^k (A_{Bi} + C_{Ti}) + \frac{Q(\lambda^n+1)}{2D(\lambda+1)} \sum_{i=1}^k rC_{Bi} D_i$$

Finally, the expected total cost function of the integrated model over the infinite planning horizon including expected manufacturers cost and customers' costs is as follows,

$$TC = TC_M + TC_B$$

For each case of the raw material orders, an updated total cost function is written independently. The total cost equations for Case 1 and Case 2 are indicated by $TC_1(Q, \lambda, n, m)$ and $TC_2(Q, \lambda, n, m)$, respectively.

4. Solution Methodology

The expected cost function has only four decision variables Q, λ, n, m . We can assume that Q, λ are continuous variables, while both n and m are discrete variables (m, n take integer values).

Case 1: When $v = \{1, 2, \dots, m\}$, the expected total relevant cost per year is given by

$$TC_1(Q, \lambda, n, m) = \frac{D}{Q}\Omega + \frac{Q}{2}r\psi + D\phi \quad (6)$$

where

$$\begin{aligned} \Omega &= \frac{(\lambda-1)}{(\lambda^{n-1})} \left[\frac{A_R}{m} + A_M + n \sum_{i=1}^k (A_{Bi} + C_{Ti}) \right] \\ \psi &= \frac{\lambda^{n-1}}{\lambda-1} \left[C_R \left(\frac{m-1}{f} + \frac{D}{Pf} \right) - C_{MP} \frac{D}{P} + (C_{MD} - C_{MP}) D \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) \right] + 2C_{MP} \frac{\lambda(\lambda^{n-1}-1)}{\lambda^2-1} + \\ &\quad 2C_{MP} \frac{D}{P} E \left[\frac{1}{1-x} \right] + \frac{\lambda^{n+1}}{D(\lambda+1)} \sum_{i=1}^k C_{Bi} D_i \\ \phi &= \frac{C_o}{f} + C_I E[x] + C_M \end{aligned}$$

Case 2: When $v = \{1, 1/2, \dots, 1/m\}$, the expected total relevant cost per year is given by

$$TC_2(Q, \lambda, n, m) = \frac{D}{Q}\delta + \frac{Q}{2}r\Gamma + D\phi \quad (7)$$

where

$$\begin{aligned} \delta &= \frac{(\lambda-1)}{(\lambda^{n-1})} [m A_R + A_M + n \sum_{i=1}^k (A_{Bi} + C_{Ti})] \\ \Gamma &= \frac{\lambda^{n-1}}{\lambda-1} \left[C_R \frac{D}{mPf} - C_{MP} \frac{D}{P} + (C_{MD} - C_{MP}) D \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) \right] + 2C_{MP} \frac{\lambda(\lambda^{n-1}-1)}{\lambda^2-1} + 2C_{MP} \frac{D}{P} E \left[\frac{1}{1-x} \right] + \\ &\quad \frac{\lambda^{n+1}}{D(\lambda+1)} \sum_{i=1}^k C_{Bi} D_i \end{aligned}$$

The problem can be formulated by

$$\begin{aligned} &\min TC(Q, \lambda, n, m) \\ &\text{Subject to } 1 < \lambda < \frac{PE[1-x]}{D} \\ &\quad Q > 0 \\ &\quad n, m \in Z^+ \end{aligned}$$

Here we first minimize expected total cost for the both cases (case 1 & case 2) and then select the case which is able to give the lower expected total cost.

Proposition 1. For fixed n, m and λ , $TC_i(Q, \lambda, n, m)$ is convex in Q . (where $i = 1, 2$)

Proof.

For fixed λ, n, m , taking first and second derivatives with respect to Q gives:

Case 1:

$$\frac{\partial TC_1}{\partial Q} = -\frac{D}{Q^2}\Omega + \frac{1}{2}r\psi, \quad \frac{\partial^2 TC_1}{\partial Q^2} = \frac{2D}{Q^3}\Omega > 0$$

Therefore, $TC_1(Q, \lambda, n, m)$ is convex on Q .

Case 2:

$$\frac{\partial TC_2}{\partial Q} = -\frac{D}{Q^2} \delta + \frac{1}{2} r \Gamma, \quad \frac{\partial^2 TC_1}{\partial Q^2} = \frac{2D}{Q^3} \delta > 0.$$

Therefore, $TC_1(Q, \lambda, n, m)$ is convex on Q . Hence the proposition follows.

Now, for optimality, setting $\frac{\partial TC_i}{\partial Q} = 0$ yields,

Case 1:

$$Q^* = \sqrt{\frac{2D\Omega}{r\psi}}. \quad (8)$$

Substituting Q^* in Eq. (6), the expected total cost function becomes

$$TC_1(Q^*, \lambda, n, m) = \sqrt{2Dr\psi\Omega} + D\varphi. \quad (9)$$

Minimization of $TC_1(Q^*, \lambda, n, m)$ is equivalent to minimization of $\sqrt{2Dr\psi\Omega}$, which is equivalent to Minimization of $2Dr\psi\Omega$, thus,

$$\min TC_1(Q^*, \lambda, n, m) \equiv \text{Min } \sqrt{2Dr\psi\Omega} \equiv \text{Min } 2Dr\psi\Omega \quad (10)$$

Case 2:

Similar to Case 1's procedure, we obtain the following functions and relationship for Case 2:

$$Q^* = \sqrt{\frac{2D\delta}{r\Gamma}} \quad (11)$$

$$TC_2(Q^*, \lambda, n, m) = \sqrt{2Dr\Gamma\delta} + D\varphi \quad (12)$$

and

$$\min TC_2(Q^*, \lambda, n, m) \equiv \text{Min } \sqrt{2Dr\Gamma\delta} \equiv \text{Min } 2Dr\Gamma\delta \quad (13)$$

Proposition 2. For fixed m , $TC_i(Q^*, n, m)$ is convex in n when $\lambda = 1$. (where = 1,2)

Proof: The proof of the proposition is straightforward and hence omitted.

Proposition 3. For fixed m , $TC_i(Q^*, n, m)$ is convex in n when $\lambda = \frac{PE[1-x]}{D} (> 1)$. (where = 1,2)

Proof.

Case 1:

Let $g(n)$ be the function of $TC_1(Q^*, n, m)$ subtracting terms which are independent of n Thus,

$$g(n) = \frac{(K+nL)}{(\lambda^n - 1)} (\lambda^n S + T) \quad (14)$$

where, $K = (\lambda - 1) \left(\frac{A_R}{m} + A_M \right)$, $L = (\lambda - 1) \sum_{i=1}^k (A_{Bi} + C_{Ti})$,

$$S = C_R \frac{1}{(\lambda-1)} \left(\frac{m-1}{f} + \frac{D}{Pf} \right) + C_{MP} \left[\frac{2}{\lambda^2-1} - \frac{D}{P(\lambda-1)} - \frac{D}{(\lambda-1)} \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) \right] + C_{MD} \frac{D}{(\lambda-1)} \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) + \frac{1}{D(\lambda+1)} \sum_{i=1}^k C_{Bi} D_i,$$

$$T = C_{MP} \left[2 \frac{D}{P} E \left[\frac{1}{1-x} \right] + \frac{D}{P(\lambda-1)} + \frac{D}{(\lambda-1)} \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) - \frac{2\lambda}{\lambda^2-1} \right] - C_R \frac{1}{(\lambda-1)} \left(\frac{m-1}{f} + \frac{D}{Pf} \right) - C_{MD} \frac{D}{(\lambda-1)} \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) + \frac{1}{D(\lambda+1)} \sum_{i=1}^k C_{Bi} D_i$$

To prove that $TC_1(Q^*, n, m)$ is convex in positive integral n , it is enough to show that $g(n)$ is convex in positive real n . We find that $g(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $g(n) \rightarrow \infty$ as $n \rightarrow 0$ (As $S > 0$). Also $g(n)$ is continuous and finite between these two limits. Thus, it is convex if it has a single turning point in the interval $(0, \infty)$.

The numerator of $g'(n)$ reduces to

$$h(n) = L(A\lambda^n + B)(\lambda^n - 1) - (S + T)\lambda^n(K + nL)\log(\lambda)$$

and the denominator is positive for $n > 0$. We therefore need to show that $h(n)$ has only one zero for positive n . $h(0) < 0$ and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$.

$$h'(n) = \lambda^n \log(\lambda) [2LS \lambda^n - 2LS - (S + T)(K + nL)\log(\lambda)]$$

$h'(n)$ is also negative when $n = 0$ and $h'(n) \rightarrow \infty$ as $n \rightarrow \infty$. Thus $h'(n) = 0$ has a single solution n^* (say). Therefore $h(n)$ is negative when $n = 0$, decreases until $n = n^*$ and then as n increases $h(n)$ increases indefinitely. Hence $h(n)$ has only one zero for positive real n and this completes the proof.

Similar to Case 1's procedure we can prove that $TC_2(Q^*, n, m)$ is convex in positive integral n .

If n' and n'' be the optimal values of n for $\lambda = 1$ and $\frac{PE[1-x]}{D}$, then following the assumption of Hill (1997), the optimal value of n for general λ lies in $[n', n'']$

Proposition 4. For fixed λ and n , $TC_i(Q^*, \lambda, n, m)$ is convex in m . (where $i = 1, 2$)

Proof:

Case 1:

Let $\theta_1(m)$ be the function of $TC_1(Q^*, \lambda, n, m)$ subtracting terms which are independent of m . Thus,

$$\theta_1(m) = \left(\frac{\alpha}{m} + \beta \right) (m\gamma + \rho) \quad (15)$$

where

$$\alpha = A_R \frac{(\lambda-1)}{(\lambda^{n-1})}, \quad \beta = n \frac{(\lambda-1)}{(\lambda^{n-1})} \sum_{i=1}^k (A_{Bi} + C_{Ti}), \quad \gamma = C_R \frac{\lambda^{n-1}}{\lambda-1} \quad \text{and}$$

$$\rho = \psi - m C_R \frac{\lambda^{n-1}}{\lambda-1}$$

To prove that $TC_1(Q^*, \lambda, n, m)$ is convex in positive integral m , it is enough to show that $\theta_1(m)$ is convex in positive real m .

$$\frac{\partial \theta}{\partial m} = \beta\gamma - \alpha\rho \frac{1}{m^2}, \quad \frac{\partial^2 \theta}{\partial m^2} = \alpha\rho \frac{1}{m^3} > 0$$

Therefore, $TC_1(Q^*, \lambda, n, m)$ is convex on m .

Case 2:

Similar to Case 1's procedure, we obtain $\theta_2(m)$ be the function of $TC_2(Q^*, \lambda, n, m)$ subtracting terms which are independent of m .

Thus

$$\theta_2(m) = \left(\frac{\eta}{m} + \sigma\right)(m\alpha + \beta), \quad (16)$$

where

$$\eta = C_R \frac{D(\lambda^n - 1)}{fP(\lambda - 1)} \quad \text{and} \quad \sigma = \Gamma - C_R \frac{D(\lambda^n - 1)}{mfP(\lambda - 1)},$$

and

$$\frac{\partial \theta}{\partial m} = \alpha\sigma - \frac{\beta\eta}{m^2}, \quad \frac{\partial^2 \theta}{\partial m^2} = \beta\eta \frac{1}{m^3} > 0.$$

Therefore, $TC_2(Q^*, \lambda, n, m)$ is convex on m .

Proposition 5. The optimal value of m^* must satisfy

$$m^*(m^* - 1) \leq \frac{\alpha\rho}{\beta\gamma} \leq m^*(m^* + 1) \quad (\text{For case 1}) \quad (17)$$

$$m^*(m^* - 1) \leq \frac{\beta\eta}{\alpha\sigma} \leq m^*(m^* + 1) \quad (\text{For case 2}) \quad (18)$$

Proof:

Case 1: We shall first assume λ, n is given and by considering m^* as the optimal value of m , according to convexity of $TC_1(Q, \lambda, n, m)$ on m , m^* will satisfy

$$\theta(\lambda, n, m^*) \leq \theta(\lambda, n, m^* - 1), \quad \theta(\lambda, n, m^*) \leq \theta(\lambda, n, m^* + 1)$$

$$\text{From Eq. (18), we obtain } m^*(m^* - 1) \leq \frac{\alpha\rho}{\beta\gamma} \leq m^*(m^* + 1)$$

$$\text{Case 2: Similar to Case 1's procedure, we obtain } m^*(m^* - 1) \leq \frac{\beta\eta}{\alpha\sigma} \leq m^*(m^* + 1)$$

This completes the proof.

Algorithmic procedure is developed as follows to obtain the optimal solution for (Q, λ, n, m) (this study adopted by Hill (1997) and Giri and Sharma (2014) to determine the optimal value of Q, λ, n).

Algorithm 1

- step 1 Substitute optimal value of Q^* obtained from Eq. (8) into Eq. (6) for case 1, Eq. (11) into Eq. (7) for case 2
- step 2 Determine the upper bound n'' of n for $\lambda = \frac{PE[1-x]}{D}$ using algorithm 2
- step 3 Initialize $n_1 = 1$
- step 4 (i) For each $\lambda \in [1, \frac{PE[1-x]}{D}]$, obtain the associate optimal value m_λ^* using the inequality Eq. (17) for case 1, Eq. (18) for case 2
 (ii) Find $TC_1(\lambda_j^*, n_j, m_{\lambda_j^*}^*) = \underset{\lambda \in [1, \frac{PE[1-x]}{D}]}{\text{Min}} TC_1(\lambda, n_j, m_\lambda^*)$
- step 5 If $n_j = n''$ then go to step 6 otherwise set $n_{j+1} = n_j + 1$ and go to step 4 to get

$$TC_1(\lambda_{j+1}^*, n_{j+1}, m_{\lambda_{j+1}^*}^*).$$

Step 6 Find $TC_1(\lambda^*, n^*, m^*) = \min_j TC_1(\lambda_j^*, n_j, m_{\lambda_j^*}^*)$ and compute the corresponding $Q^*(\lambda^*, n^*, m^*)$ from Eq. (10), then $(Q^*, \lambda^*, n^*, m^*)$ is the optimal solution and $TC_1(Q^*, \lambda^*, n^*, m^*)$ is the minimum expected total cost for case 1.

To determine the upper bound n'' of n for $\lambda = \frac{PE[1-x]}{D}$ we use the following algorithm

Algorithm 2

- step 1 Substitute optimal value of Q^* obtained from Eq. (8) into Eq. (6) for case 1
- step 2 Initialize $n = 1$, $\lambda = \frac{PE[1-x]}{D}$ and $TC_1(0, m_0) = \infty$.
- step 3 Determine the associate optimal value m_n using the inequality Eq. (17) for case 1 and compute the corresponding $TC_1(n, m_n)$.
- step 4 If $TC_1(n, m_n) < TC_1(n-1, m_{n-1})$ then Set $n = n + 1$ and go to step 2 to get $TC_1(n+1, m_{n+1})$ otherwise go to step 5
- step 5 Set $n - 1 = n''$.

Similar to the case 1, using the algorithm 1 we evaluate the optimal solution for the case 2 and then select the case which is able to give the lower expected total cost.

Equal-sized Shipments

For the equal-sized shipments ($\lambda = 1$) from manufacturer to buyers the expected total cost function for both cases (using L'Hospital's Rule) reduces to

For Case 1:

$$TC_1 = \frac{D}{Q} \left[\frac{1}{n} \left(\frac{A_R}{m} + A_M \right) + \sum_{i=1}^k (A_{Bi} + C_{Ti}) \right] + \frac{Q}{2} r \left[n C_R \left(\frac{m-1}{f} + \frac{D}{fP} \right) + C_{MP} \left[n \left(1 - \frac{D}{P} \right) + \frac{2D}{P} E \left[\frac{1}{1-x} \right] - 1 \right] + n(C_{MD} - C_{MP})D \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) + \frac{1}{D} \sum_{i=1}^k C_{Bi} D_i \right] + D \left[\frac{C_o}{f} + C_I E[x] + C_M \right] \quad (19)$$

For Case 2:

$$TC_2 = \frac{D}{Q} \left[\frac{1}{n} (m A_R + A_M) + \sum_{i=1}^k (A_{Bi} + C_{Ti}) \right] + \frac{Q}{2} r \left[n C_R \frac{D}{mfP} + C_{MP} \left[n \left(1 - \frac{D}{P} \right) + \frac{2D}{P} E \left[\frac{1}{1-x} \right] - 1 \right] + n(C_{MD} - C_{MP})D \left(\frac{E[x]}{P} + \frac{E[x^2]}{P_1} \right) + \frac{1}{D} \sum_{i=1}^k C_{Bi} D_i \right] + D \left[\frac{C_o}{f} + C_I E[x] + C_M \right] \quad (20)$$

5. Numerical Examples

Let us consider a numerical example of supplying an item to 5 buyers by a manufacturer. The data of buyers are given in Table 1. Manufacturer's production rate is $P = 31700$ and total demand $D = \sum_{i=1}^k D_i = 12600$. The defective rate x is uniformly distributed with probability density function:

$$f(x) = \begin{cases} \frac{1}{0.3} & \text{if } 0 \leq x \leq 0.3 \\ 0 & \text{otherwise} \end{cases}$$

Table 1

Data for a single-manufacturer 5-buyer problem

i^{th} Buyer	A_{Bi}	D_i	C_{Ti}	C_{Bi}
1	11	3000	15	44
2	10.5	2300	10	46
3	12	1750	18	39
4	9.5	2900	15	47
5	8	2650	15	43

All defective items produced are considered to be repairable and reworked at a rate of $P_1 = 33000$.

The other parameters are: $r = 0.2, A_R = 100, A_M = 750, C_R = 10, C_{MP} = 24, C_{MD} = 15,$

$C_I = 8, f = 0.8, C_o = 20, C_M = 18$. Based on the above numerical data and using the algorithm 1 (for unequal shipments), we obtain the optimal results as given in Table 2.

Table 2

Optimal results for unequal shipment

Case	n	λ	m	TC
1	2	1.761111	1	570924.7963
2	2	1.761111	1	570924.7963
1	3	1.645868	1	570410.4626
2	3	1.645868	1/2	570348.8303
1	4	1.437453	1	570394.8257
2	4	1.437453	1/2	570222.4789
1	5	1.332380	1	570563.1776
2	5	1.332380	1/2	570304.3729
1	6	1.268477	1	570817.6924
2	6	1.268477	1/2	570486.6199
1	7	1.225359	1	571116.2496
2	7	1.225359	1/2	570722.2437

Fig 2 shows that expected total cost per year for the both cases 1 and 2 is strictly convex function on n . As a result, we are sure that the minimum obtained from the proposed algorithm is indeed the global optimum solution. From Table 2, we observe that optimal solution is obtained in case 2, when $n^* = 4, v^* = 1/2, \lambda^* = 1.437453$ and the corresponding minimum expected total cost per year $TC^* = TC_2^* = 570222.4789$, when $\lambda = 1$ we obtain $n^* = 4, v^* = 1/2$ and the corresponding minimum expected total cost per year $TC_2^* = 570833.5676$.

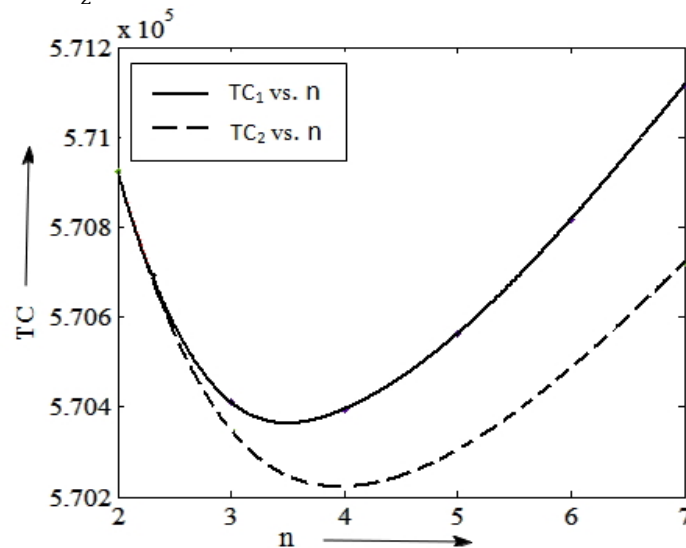
**Fig. 2.** The expected total profit for various value of n for Cases 1 and 2

Table 3

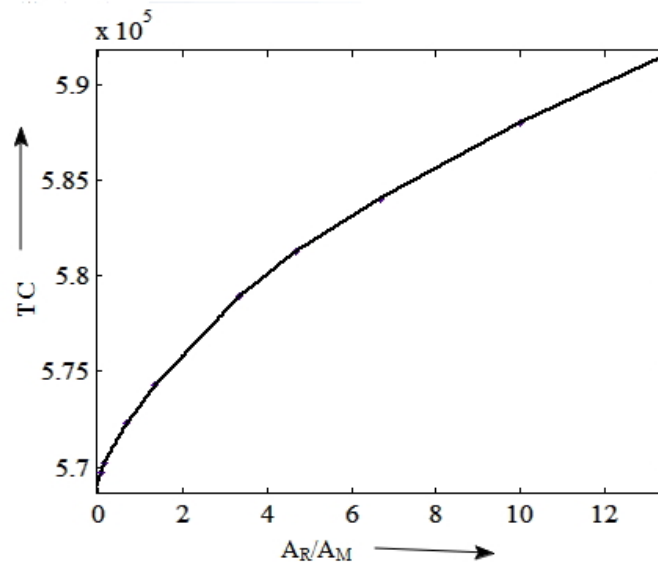
A comparative study of the results for a single manufacturer 5-buyer problem

Component	Model : I	Model : II
	Unequal sized shipment to buyer	Equal sized shipment to buyer
Q_R, v	1712.049, 1/2	1636.856, 1/2
Q_M, n	2739.278, 4	2618.968, 4
S_i	366.513, 526.845, 757.315, 1088.605	654.742, 654.742, 654.742, 654.742
TC_M	564442.6855	565556.5461
TC_{Bi}	1308.4619, 1042.5075, 981.1632, 1307.9041, 1139.7567	1186.2700, 944.2820, 931.9785 , 1179.7481, 1034.7429
TC	570222.4789	570833.5676

A comparative study of the results of the Model I and Model II is given in Table 3. The cost reduction of Model I over Model II is about 11%. Observe that the inventory cost of the manufacturer obtained in Model II is about 1.002 times higher than that of obtained in Model I but the total cost for each buyer in Model I is higher than the corresponding cost in Model II, so the cost reduction by Model I over Model II is mainly due to cost reduction in the cost of the manufacturer.

Table 4Optimal results for different A_R

A_R	v^*	Q_R^*	TC^*
50	1/2	1651.789	569754.2638
1000	1	4309.510	574243.3165
10000	3	12894.301	591389.8670

**Fig. 3.** Impact of AR/AM on total cost

A_R plays a significant role in the optimal solution. Table 4 shows how the solution changes due to change of the value of A_R from \$50/order to \$10000/order while all other parameters remain unchanged. The increase in the ordering cost causes the increase of the optimal order size of the raw material Q_R to reduce the number of raw material orders. The change of A_R is expressed using the ratio of raw material ordering cost versus set up cost A_R/A_M for the simplicity of expression. Fig. 3 shows the impact of the ratio increase, the increase of raw material ordering cost, on the expected total cost of the integrated inventory system.

6. Conclusion

In this paper we have developed an integrated production-delivery inventory model with imperfect production process in a supply chain consisting of a single manufacturer and multi-buyer. We have considered all types of costs such as fixed, material and holding costs at each stages in the supply chain: raw material procurement from supplier, manufacturing and remanufacturing at manufacturer’s facility and purchasing by customers. The effective algorithms have been developed to obtain an optimal set of lot sizes and numbers of shipments which will minimize the expected total cost. We have observed that the model I when successive shipments to buyer are increasing by a constant factor gives improved result than the Model II under equal shipment policy.

Appendix A.

Case 1: In this case $v = \{1,2,\dots,m\}$ i.e. each lot size of ordered raw material will meet the demand of m (say) production run. Therefore, for this policy $Q_R = mQ_M/f$ and the cycle length is mQ_M/D . From Fig. 1, the stock holding area for raw material is

$$\frac{mQ_M^2}{2fP} + \frac{Q_M^2}{fD} [(m - 1) + (m - 2) + \dots + 1] \tag{A.1}$$

Thus, the average inventory for raw material per year (Q_{Ravg}) is (A.1) divided by cycle length. Therefore

$$Q_{Ravg} = \frac{Q(\lambda^n - 1)}{2f(\lambda - 1)} \left(\frac{D}{P} + m - 1 \right) \tag{A.2}$$

Case 2: In this case $v = \{1,1/2,\dots,1/m\}$ i.e. manufacturer needs to replenish raw material m (say) times for every production run. Therefore, for this policy $Q_R = Q_M/mf$ and the cycle length is Q_M/D . From Fig. 2, the stock holding area for raw material is

$$m \frac{Q_M^2}{2m^2 fP} \tag{A.3}$$

Thus, the average inventory for raw material per year (Q_{Ravg}) is (A.3) divided by cycle length. Hence,

$$Q_{Ravg} = \frac{Q(\lambda^n - 1)}{2f(\lambda - 1)} \left(\frac{D}{mP} \right) \tag{A.4}$$

Appendix B

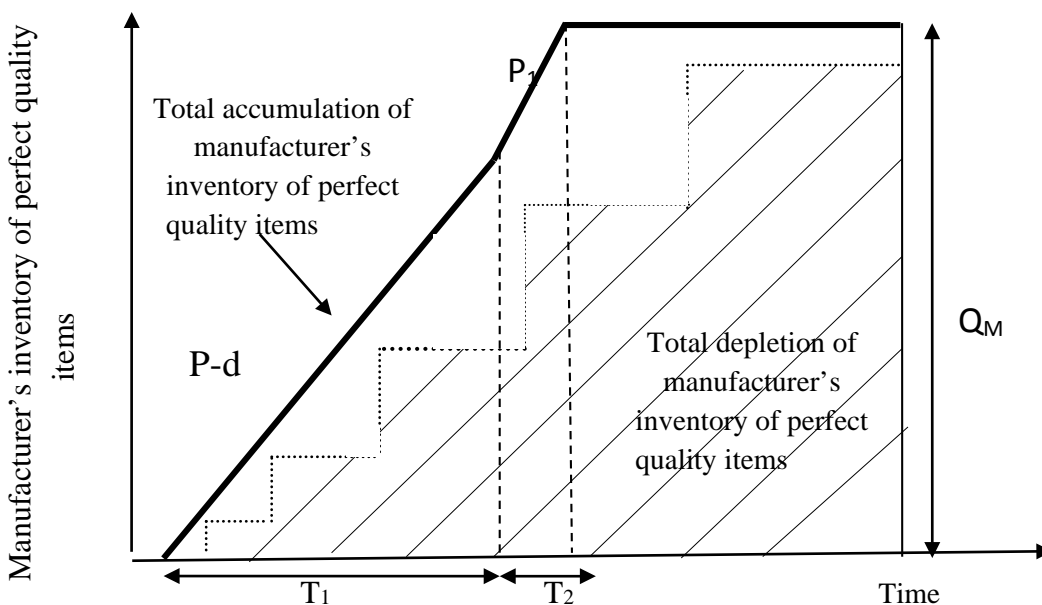


Fig. 4. Accumulation and depletion of manufacturer's inventory of perfect quality items

From Fig 4, the stock holding area for the manufacturer finished item of perfect quality is as follows,

= bold area – shaded

$$\begin{aligned}
 &= \frac{1}{2} \frac{Q_M}{P} (1-x) Q_M + \frac{x^2 Q_M^2}{2P_1} + (1-x) Q_M^2 \frac{x}{P_1} + Q_M \left[\frac{Q}{P(1-x)} + \frac{Q}{D} + \frac{\lambda Q}{D} + \dots + \frac{\lambda^{n-2} Q}{D} - \frac{Q_M}{P} - \frac{x Q_M}{P_1} \right] - \\
 &\left[\frac{Q^2}{D} + (Q + \lambda Q) \frac{\lambda Q}{D} + \dots + (Q + \lambda Q + \dots + \lambda^{n-2} Q) \frac{\lambda^{n-2} Q}{D} \right] \\
 &= \frac{1}{2} \frac{Q_M}{D} \left[\frac{2DQ}{P(1-x)} + Q \frac{2\lambda(\lambda^{n-1}-1)}{\lambda^2-1} - Q \frac{\lambda^{n-1}}{\lambda-1} \left[\frac{D}{P} + D \left(\frac{x}{P} + \frac{x^2}{P_1} \right) \right] \right] \quad (B.1)
 \end{aligned}$$

The cycle length is Q_M/D year. Thus, the average inventory for manufacturer finished item of perfect quality per year (Q_{Pavg}) is (B.1) divided by the cycle length. Therefore

$$Q_{Pavg} = \frac{Q}{2} \left[\frac{2D}{P(1-x)} + \frac{2\lambda(\lambda^{n-1}-1)}{\lambda^2-1} - \frac{\lambda^{n-1}}{\lambda-1} \left[\frac{D}{P} + D \left(\frac{x}{P} + \frac{x^2}{P_1} \right) \right] \right] \quad (B.2)$$

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