Uncertain Supply Chain Management 7 (2019) 33–48

Contents lists available at GrowingScience

Uncertain Supply Chain Management

homepage: www.GrowingScience.com/uscm

An inventory model with credit, price and marketing dependent demand under permitted delayed payments and shortages: A signomial geometric programming approach

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C H R O N I C L E

Article history:
Received February 2, 2018
Accepted May 23 2018
Available online
May 29 2018

Keywords:
Credit-dependent demand
Partial delayed payment
Partial backordering
Signomial geometric programming

A B S T R A C T

In this study, we incorporate trade credit policy into a joint marketing and pricing problem in which demand rate depends on the length of the credit period provided by the retailer for her customers, marketing expenditure, and selling price. The trade credit policy adopted here is a delayed payment policy in partial form in which the customers must pay a percent of the total purchasing cost at the time of placing an order and can pay the remaining amount later. Shortages are allowed and partially backordered. The main objective of this study is to determine the optimal credit period, marketing expenditure, selling price, and variables of inventory control simultaneously in order to maximize retailer's total profit. For solving the proposed problem, first an approximation method is applied to simplify the profit function and transform the problem into a constrained Signomial Geometric Programming (SGP) problem, then a global optimization approach is used for solving the model. Finally, a numerical example and sensitivity analysis of the important parameters are conducted to show the effectiveness of proposed approach.

1. Introduction

In classic economic order quantity (EOQ) model, it is assumed that marketing strategies and production are executed, separately. However, these two factors are inextricably interdependent. In this regard, coordination of marketing strategies and production has an absolutely essential role in profit maximization in competitive business world. The first study considered a model incorporating production and marketing strategies was performed by Lee and Kim (1993). They assumed demand as non-deterministic and expressed it as a power function of selling price and marketing expenditure. The paper aimed to determine the marketing expenditure, selling price, demand and the order quantity in a net profit – maximizing. After that, several researchers considered this assumption in their models (Bayati et al., 2013; Sadjadi et al., 2010; Sadjadi et al., 2005; Samadi et al., 2013; Tabatabaei et al., 2017). In today's business transaction, it is very common to observe the customers who are not willing to pay immediately after buying the goods or services and are allowed to delay their payments till the end of the credit period. The customer pays no interest during the constant and predetermined period of time in which they have to settle the account, but if the payment is delayed after the period, interest

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doi: 10.5267/j.uscm.2018.5.004

will be charged. During permissible period the customer can sell or use the goods and keep on revenue accumulation. Therefore, it is beneficial for the customer to postpone the payment to the supplier until the end of the permissible period. Goyal (1985) was the first person who considered an allowed delay in payment for customer in his model with general presumption of classic EOQ model. Afterwards, Liao et al. (2000) explored a model for initial-stock-dependent consumption rate by considering delay in payments. In their proposed model, shortages were not permissible. They also investigated the effect of initial-stock-dependent consumption, inflation, deterioration rates, and delay on payment. Teng (2002) modified the model discussed in Goyal (1985) by considering the distinction among unit price and unit cost. Shinn and Hwang (2003) presented an EOQ model in which demand rate depends on the selling price and credit period and credit period depends on the order quantity. Ho (2011) developed a new mathematical formulation under two level of trade credit policy in which demand is sensitive to the credit period offered by the retailer and selling price. Furthermore, many researches were studied on this filed by considering different assumptions for payments (Ghoreishi et al., 2015; Jaggi et al., 2015; Khanna et al., 2017; Sharma, 2016; Taleizadeh et al., 2013).

However, all the aforementioned studies mainly take the retailer's perspective of obtaining the optimal ordering policies under a predetermined delay period, but little is known about how to find the optimal length of the delay period offered by the retailer to the customers. On the other hand, since a permissible delay in payment leads to bring new customers and increase demand rate. Thus, in real-life situations it is necessary to study the effect of delay period on the demand rate. In addition, an effective way to show the effect of delay period on the demand rate and find the optimal delay period is to represent the demand rate as power function of the length of delay period offered by the retailer, which is the first main component of this paper.

The second main component of the proposed model is partial backlogging of demand. Shortages are very significant, especially in an inventory model in which delay in payment is accepted, because shortages can affect the order quantity to make more profit from the delay in payments (Jamal et al., 1997). Montgomery et al. (1973) were the first who developed an inventory model by general presumptions of the classic EOQ model under partial backlogging of demand. Nowadays, many researchers consider shortages in their models, especially with different forms of payments such as Tripathi (2012), Taleizadeh et al. (2013), Lashgari et al. (2016), Diabat et al. (2017), and Cunha et al. (2018). For these studies, all assumed the fraction of shortage backordered is constant over time. This assumption is not valid in the real markets. Usually, the backorder rate depends on the length of the waiting time for the next replenishment and is a decreasing function of the waiting time. This assumption is considered in the few studies such as Maihami and Abadi (2012)¸ Dye and Yang (2015), Sharma (2016) , and Maihami et al. (2017). The third main component of the proposed model is to apply appropriate approach for finding global optimal solutions. As an aforementioned, the demand rate and unit cost are not constant and are represented as multivariate functions of different parameters. These assumptions convert the model to a nonlinear programing problem. According to the literature, to solve these kinds of nonlinear problems in inventory models Geometric Programing (GP) method is applied frequently (El-Wakeel & Al Salman, 2018; Mandal, 2016; Sadjadi et al., 2010; Samadi et al., 2013; Tabatabaei et al., 2017). GP problem was discovered by Zener (1971) for solving engineering problems where objective functions were positive sums of log-linear functions. Signomial Geometric Programming (SGP) problems were the first extension of GP problems that includes Signomial expressions in the objective function and constraints (Passy & Wilde, 1967). This method has very useful computational and theoretical properties to solve complex optimization problems in different fields such as engineering, management, science, etc. This technique was extended rapidly by researchers, especially engineering designers.

Table 1
The comparison table of related studies

Studies	Demand	Unit cost	Delayed payment		Backordering		Backorder rate	Decision variables			Solved by GP method
			Full	Partial	Full	Partial		Credit period	price	Marketing expenditure	
Ho (2011)	Price- credit period -linked	Constant	Yes	No	No	No	-	No	Yes	No	No
Taleizadeh et al. (2013)	Constant	Constant	No	Yes	No	Yes	Constant	No	No	No	No
Samadi et al. (2013)	Price- marketing and service expenditures - linked	Order quantity - linked	No	No	Yes	No	Constant	No	Yes	Yes	Yes
Dye and Yang (2015)	Credit period- linked	Constant	Yes	No	No	Yes	Waiting time- dependent	Yes	No	No	No
Jaggi et al. (2015)	Price-linked	Constant	Yes	No	No	No	-	No	No	No	No
Sharma (2016)	Constant	Constant	No	yes	No	Yes	Waiting time- dependent	No	No	No	No
Lashgari et al. (2016)	Constant	Constant	Yes	No	Yes	No	-	No	No	No	No
Maihami et al. (2017)	Price-linked	Constant	No	Yes	No	Partial	Waiting time- dependent	No	Yes	No	No
Tabatabaei et al. (2017)	Price- marketing expenditures - linked	Order quantity - linked	No	No	No	No	-	No	Yes	Yes	Yes
This study	Credit period- price- marketing expenditures - linked	Order quantity - linked	No	Yes	No	Yes	Waiting time- dependent	Yes	Yes	Yes	Yes

The comparison table of related studies is given in Table 1. According to this table, demand is linked to different parameters such as marketing expenditure, selling price, service expenditure, time, and delay period. To the best of our knowledge, none of researchers has considered the effect of the length of credit period offered by the retailer to its customers, marketing expenditure and selling price on demand simultaneously in one model. For the first time, we propose a new inventory model under partial delayed payments that considers demand as multivariate function of the credit period, marketing expenditure, and selling price. In addition, in order to make the model more practical, it is assumed that shortages are allowed and partially backordered in which the backorder rate is variable and depends on the waiting time for the next replenishment. The unit purchasing cost is linked the order quantity. The main objective of this study is to determine the optimal credit period, marketing expenditure, selling price, and variables of inventory control simultaneously in order to maximize retailer's total profit. For solving the proposed problem, a global optimization of SGP problems is applied. The proposed model is based on the real constraints and environments of manufacturing firms and suppliers such as automotive supplier firms and drug manufacturers.

Given the literature, it is well-known that SGP problems are non-convex class of problems and an inherently intractable NP-hard problem (Xu, 2014), for the reason finding a global optimal solution for SGP problems are roughly difficult. In this technique degree of difficulty (DD = The number of decision variables + the numbers of terms in objective functions and constraints -1) has an important role. When $DD \leq 2$, many researchers have applied dual geometric programming for solving inventory models. But if $DD \geq 3$, solving inventory models will be difficult. Since the important section SGP problem is the method used. Over the past decade, several researchers have considered this issue with interest for finding global optimization strategies for these kinds of problems. In this study, we apply the proposed approach by Xu (2014) in order to solve the presented model. This approach transforms the non-convex SGP into series of standard GP problem to obtain a global solution.

The rest of this paper is organized as follows. We first describe the problem definition in Section 2. Section 3 provides notations and model formulation. Then, the proposed model is solved using global optimization approach in Section 4. In Section 5, numerical examples are conducted and also sensitivity analysis of important parameters are presented. Finally, conclusion remarks and future works are provided in Section 6.

2. Problem definition

Consider a supply chain consisting of the retailer and customers. In order to motivate customers and also reduce default risks with credit-risk customers, the retailer offers a partial delayed payment for credit-risk customers who must pay the η percent of the total purchased amount at the time of receiving items and then obtain a delay period of M years on the remaining amount. The demand rate is sensitive to selling price, credit period offered by the retailer, and marketing expenditure. Shortages are allowed and partially backordered.

We also consider the following assumptions in our problem:

- Demand rate (λ) can be considered as a power function of credit period (M), marketing expenditure (G), and selling price (S) by related elasticities that is determined by following equation:
  $$\lambda = VM^{\delta}G^{\chi}S^{-\alpha} \tag{1}$$
  where V is marketing size and $\delta > 0$, $\chi > 0$, $\alpha \geq 1$ are the credit period, marketing expenditure and selling price elasticity, respectively.
- The unit purchasing cost (P_r) is a decreasing function of the order quantity (Q) as follows:
  $$P_r = UQ^{-\gamma} \tag{2}$$
  where U and $\gamma > 0$ are scaling coefficient and discount coefficient, respectively.

- The rate of backordered demand is a function of waiting time length for next replenishment, i.e. $\beta(t)=e^{-\varepsilon t}$, where t is the length of waiting time for next replenishment and $\varepsilon \geq 0$ is backlogging parameter.
- The time horizon is infinite, the lead time is zero and the replenishment rate is instantaneous.
- There is no deterioration.
- In offering delayed payment to customers, the retailer endures a capital opportunity cost at rate I_p.
- All parameters are supposed precise and constant.

3. Problem formulation

To formulate the problem, first the notations are introduced in Section3.1. Then the inventory model is developed in Section3.2.

3.1. Notation

Parameters :	
A	Ordering cost (\$/order)
h	Holding cost (\$/unit/year)
π	Backorder cost (\$/unit/year)
π_l	The cost of goodwill lost (\$/unit)
I_p	Rate of opportunity cost (\$/year)
α	Selling price elasticity to demand
χ	Marketing expenditure elasticity to demand
δ	Credit period elasticity to demand
γ	discount coefficient (the order quantity elasticity to unit cost)
η	The portion of the purchase cost that should be paid when an order is placed (initial payment), $\eta \in [0,1]$
Decision variables:	
P	The portion of demand that will be satisfied from stock, $P \in [0,1]$
T	The length of an inventory cycle time
S	The unit selling price
M	Credit period
G	Marketing expenditure per unit item
Independent decision variable:	
λ	Demand rate per year
$I(t)$	The inventory level at time t
P_r	Unit purchasing cost (\$/unit)
$\beta(t)$	The fraction of shortages that will be backordered, $1<\beta(t)<1$, $\beta(0)=1$
Q	Order quantity
B	The maximum level of backordered demand
Z	Average annual retailer's profit

3.2. The mathematical model

In the starting of each inventory cycle, the retailer orders Q units and offers a partial credit period of M years to its customers. During the time interval $[0. PT]$, the inventory level is declining due to demand and reaches zero at time PT. Finally, shortages happen during the time interval $[PT. T]$. Fig 1 shows the described inventory system. The main goal of the inventory problem here is to obtain the best amount of credit period, selling price, marketing expenditure, and replenishment decisions so that the retailer's total profit is maximized. According to Fig 1 and above description, the following differential equations represent the change of inventory level at any time:

$$\frac{dI(t)}{dt} = \begin{cases} -\lambda & \text{if } \; 0 \le t \le PT \\ -\lambda\beta(T-t) & \text{if } \; PT \le t \le T \end{cases} \tag{3}$$

By the boundary condition $I_1(PT) = I_2(PT) = 0$ (see Fig. 1), the solution of Eq. (1) is:

$$I(t) = \begin{cases} I_1(t) & \text{if } \; 0 \le t \le PT \\ I_2(t) & \text{if } \; PT \le t \le T \end{cases} \tag{4}$$

where

$$I_1(t) = \lambda(PT - t) \tag{5}$$

$$I_2(t) = -\frac{\lambda}{\varepsilon}\left(e^{-\varepsilon(T-t)} - e^{-\varepsilon T(1-P)}\right) \tag{6}$$

Putting $t = T$ into Eq. (6), the maximum level of backordered demand per cycle is determined as follows:

$$B = -I_2(T) = \frac{\lambda}{\varepsilon}\left(1 - e^{-\varepsilon T(1-P)}\right). \tag{7}$$

Therefore, the order quantity per cycle is the sum of initial inventory on hand and the number of backorders as follow:

$$Q = I_1(0) + B = \lambda\left(PT + \left(1 - e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right) \tag{8}$$

According to Ho (2011), total profit function per year is calculated by following conceptual formulation.

$$\text{Total profit per year} = \left\{\begin{array}{l} \text{sales revenue} - \text{purchasing cost} - \text{marketing expenditure} \\ - \text{fixed ordering cost} - \text{holding cost} - \text{shortages cost} \\ - \text{opportunity cost} \end{array}\right\} \text{per year}$$

Therefore, the components of the retailer's total annual profit can be calculated as follows:
Sales revenue: the average annual revenue from sale (S_r) is calculated as follows:

$$S_r = \frac{SQ}{T} = \frac{S\lambda\left(PT + \left(1 - e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)}{T} = VM^{\delta}G^{\chi}S^{1-\alpha}\left(PT + \left(1 - e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)T^{-1} \tag{9}$$

Purchasing cost: according to Eqs. (1-3), the average annual purchasing cost (C_p) is calculated as follows:

$$C_p = \frac{P_r Q}{T} = \frac{UQ^{1-\gamma}}{T} = \frac{U\left(\lambda\left(PT + \left(1 - e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)\right)^{1-\gamma}}{T} = U\lambda^{1-\gamma}T^{-1}\left(PT + \left(1 - e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)^{1-\gamma}$$

$$= UV^{1-\gamma}M^{\delta(1-\gamma)}G^{\chi(1-\gamma)}S^{-\alpha(1-\gamma)}T^{-1}\left(PT + \left(1 - e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)^{1-\gamma} \tag{10}$$

Marketing expenditure: average marketing expenditure per year (C_m) is calculated by the following equations:

$$C_m = \frac{GQ}{T} = \frac{G\lambda\left(PT + \left(1-e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)}{T} = G\lambda\left(PT + \left(1-e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)T^{-1}$$
$$= VM^{\delta}G^{\chi+1}S^{-\alpha}\left(PT + \left(1-e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right)T^{-1} \quad (11)$$

Fixed ordering cost: retailer's ordering cost per cycle is constant and equal to A , so the ordering cost of the retailer per year (C_f) is calculated as follows:

$$C_f = \frac{A}{T} \quad (12)$$

Holding cost: referring to Fig1., the average inventory holding cost per year (C_h) is given by:

$$C_h = \frac{1}{T}\left(h\frac{\lambda PT \times PT}{2}\right) = 0.5h\lambda P^2T = 0.5hVM^{\delta}G^{\chi}S^{-\alpha}P^2T \quad . \quad (13)$$

Shortage cost: as previously mentioned, system confronts a partial backorder during the time $[PT.T]$. Since, the average backorders cost (C_{sh}) and average goodwill cost (C_{gl}) for lost sales in year are calculated by Eq. (14) and Eq. (15), respectively (see Fig 1).

$$C_{sh} = \frac{1}{T}\left(\pi\int_{PT}^{T}\left(-I_2(t)\right)dt\right) = \lambda\pi\left(\frac{1-e^{-\varepsilon T(1-P)}}{\varepsilon^2} - \frac{T(1-P)e^{-\varepsilon T(1-P)}}{\varepsilon}\right)T^{-1}$$
$$= \pi VM^{\delta}G^{\chi}S^{-\alpha}\left(\frac{1-e^{-\varepsilon T(1-P)}}{\varepsilon^2} - \frac{T(1-P)e^{-\varepsilon T(1-P)}}{\varepsilon}\right)T^{-1} \quad (14)$$

$$C_{gl} = \frac{1}{T}\left(\pi_l\int_{PT}^{T}\lambda\left(1-\beta(T-t)\right)dt\right) = \lambda\pi_l\left(T - PT - \frac{1-e^{-\varepsilon T(1-P)}}{\varepsilon}\right)T^{-1}$$
$$= \pi_l VM^{\delta}G^{\chi}S^{-\alpha}\left(T - PT - \frac{1-e^{-\varepsilon T(1-P)}}{\varepsilon}\right)T^{-1} \quad (15)$$

Opportunity cost: providing a delay period M to the customers, the retailer endures a capital opportunity cost with a finance rate I_p for the $(1-\eta)$ percent of the total purchasing cost. Since, the average annual opportunity cost (C_{op}) is calculated as follows:

$$C_{op} = \frac{1}{T}\left(I_p(1-\eta)SQM\right) = I_p(1-\eta)VM^{\delta+1}G^{\chi}S^{1-\alpha}\left(PT + \left(1-e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1}\right) \quad (16)$$

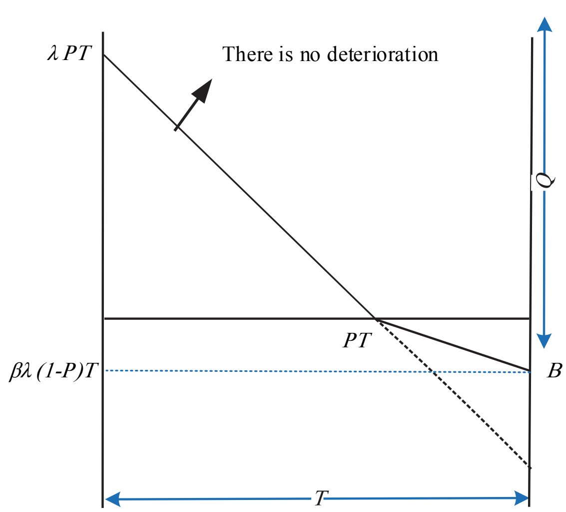

Fig. 1. Behavior of inventory system

Therefore, under credit period-selling price-marketing expenditure dependent demand, partial delayed payment, and partial backordering with time-dependend backorder rate, the objective of this research is to obtain the order quantity, credit period offered by the retailer to its customers, replenishment time, selling price, marketing expenditure, and the portion of demand that will be satisfied from stock to maximize the average retailer's profit (Z). So, the mathematical model of the inventory problem can be defined as follows:

$$\max Z(x) = S_r - C_p - C_m - C_f - C_h - C_{sh} - C_{gl} - C_{op}$$
$$\text{subject to} \qquad x = (S, M, G, T, P) \geq 0 \tag{17}$$

4. Solution methodology

The number of decision variables and the exponential terms of the total profit function make the problem more difficult to solve. So, for solving the proposed problem, first a truncated Taylor series expansion for approximating the exponential terms is applied; then, the proposed problem will be written as a signomial geometric programming (SGP) problem. Since the signomial geometric programming problems belong to a nonconvex class of problems that is an intrinsically intractable NP-hard problem, these problems are hard to solve for global optimality. Therefore, we apply the global optimization approach is proposed by Xu (2014) to obtain the optimal solutions and the corresponding maximum profit. In this approach, first some convexification and conversion strategies are used for transforming the basic SGP problems into a series of standard GP problems that are nonlinear convex problems and can be efficiently solved, then the proposed approach is presented as an iterative algorithm to obtain the global optimum solutions.

Eq. (17) is transformed into the following problem, after using the first three terms of a truncated Taylor series expansion of the exponential terms and defining an additional constrain and variable:

$$\begin{aligned} Max\ Z(x) = {} & VM^{\delta}G^{\chi}S^{1-\alpha}NT^{-1} - UV^{1-\gamma}M^{\delta(1-\gamma)}G^{\chi(1-\gamma)}S^{-\alpha(1-\gamma)}T^{-1}N^{1-\gamma} - VM^{\delta}G^{\chi+1}S^{-\alpha}NT^{-1} \\ & -AT^{-1} - VM^{\delta}G^{\chi}S^{-\alpha}\left(0.5hP^{2}T + 0.5\pi T - \pi TP + 0.5\pi TP^{2} - 0.5\pi\varepsilon T^{2}\right. \\ & \left. +1.5\pi\varepsilon T^{2}P - 1.5\pi\varepsilon T^{2}P^{2} + 0.5\pi\varepsilon TP^{3} + 0.5\pi_{l}\varepsilon T + 0.5\pi_{l}\varepsilon TP^{2} - \pi_{l}\varepsilon TP\right) \\ & -I_{p}(1-\eta)VM^{\delta+1}G^{\chi}S^{1-\alpha}N \end{aligned} \tag{18}$$

subject to

$$PT + \left(1 - e^{-\varepsilon T(1-P)}\right)\varepsilon^{-1} = N \xrightarrow{\text{truncated Taylor series expansion}} T - 0.5\varepsilon T^{2} - 0.5\varepsilon T^{2}P^{2} + \varepsilon T^{2}P = N \tag{19}$$

$$x = (S, M, G, T, P) \geq 0 \tag{20}$$

Now, we can convert Eq. (18-20) to a constrained SGP problem as follows by using the general form of constrained SGP that is given in Appendix:

$$\begin{aligned} Min\ Z'(x) = {} & -VM^{\delta}G^{\chi}S^{1-\alpha}NT^{-1} + UV^{1-\gamma}M^{\delta(1-\gamma)}G^{\chi(1-\gamma)}S^{-\alpha(1-\gamma)}T^{-1}N^{1-\gamma} \\ & +VM^{\delta}G^{\chi+1}S^{-\alpha}NT^{-1} + AT^{-1} + VM^{\delta}G^{\chi}S^{-\alpha}\left(0.5hP^{2}T + 0.5\pi T - \pi TP\right. \\ & \left. -0.5\pi\varepsilon T^{2} + 1.5\pi\varepsilon T^{2}P - 1.5\pi\varepsilon T^{2}P^{2} + 0.5\pi\varepsilon TP^{3} + 0.5\pi_{l}\varepsilon T + 0.5\pi_{l}\varepsilon TP^{2}\right) \end{aligned}$$

$$+0.5\pi_l\,\varepsilon TP^2 - \pi_l\,\varepsilon TP\Big) + I_p\,(1-\eta)VM^{\delta+1}G^{\chi}S^{1-\alpha}N \tag{21}$$

subject to $$TN^{-1} - 0.5\varepsilon T^2N^{-1} - 0.5\varepsilon T^2P^2N^{-1} + \varepsilon T^2PN^{-1} = 1 \tag{22}$$

$$x = (S, M, G, T, P) \geq 0 \tag{23}$$

It should be noted that the objective function derived from the model (21) is the reciprocal of the profit Z.

As expressed in the proposed approach of Xu (2014), we first rewrite the above problem as:

$$\min Z'(x) = Z'^{+}(x) - Z'^{-}(x) \tag{24}$$

subject to $$\kappa^{+}(x) - \kappa^{-}(x) = 1 \tag{25}$$

$$x = (S, M, G, T, P) \geq 0 \tag{26}$$

where $Z'^{+}(x)$ and $Z'^{-}(x)$ are positive and negative terms of objective function (21) respectively , $\kappa^{+}(x)$ and $\kappa^{-}(x)$ are positive and negative terms of constraint (22) respectively that are calculated as:

$$\begin{aligned} Z'^{+}(x) = {} & UV^{1-\gamma}M^{\delta(1-\gamma)}G^{\chi(1-\gamma)}S^{-\alpha(1-\gamma)}T^{-1}N^{1-\gamma} + VM^{\delta}G^{\chi+1}S^{-\alpha}NT^{-1} \\ & + AT^{-1} + VM^{\delta}G^{\chi}S^{-\alpha}\Big(0.5hP^2T + 0.5\pi T + 0.5\pi TP^2 + 1.5\pi\varepsilon T^2P \\ & + 0.5\pi\varepsilon TP^3 + 0.5\pi_l\,\varepsilon T + 0.5\pi_l\,\varepsilon TP^2\Big) + I_p\,(1-\eta)VM^{\delta+1}G^{\chi}S^{1-\alpha}N \end{aligned} \tag{27}$$

$$\begin{aligned} Z'^{-}(x) = {} & VM^{\delta}G^{\chi}S^{1-\alpha}NT^{-1} + VM^{\delta}G^{\chi}S^{-\alpha}\Big(\pi TP + 0.5\pi\varepsilon T^2 + 0.5\pi\varepsilon TP^3 \\ & + 0.5\pi_l\,\varepsilon T + 0.5\pi_l\,\varepsilon TP^2 + 1.5\pi\varepsilon T^2P^2 + \pi_l\,\varepsilon TP\Big) \end{aligned} \tag{28}$$

$$\kappa^{+}(x) = TN^{-1} + \varepsilon T^2PN^{-1} \tag{29}$$

$$\kappa^{-}(x) = 0.5\varepsilon T^2N^{-1} + 0.5\varepsilon T^2P^2N^{-1} \tag{30}$$

Then, the SGP problem (21) – (26) can be transformed into the following forms:

$$\min\; Z'^{+}(x) - Z'^{-}(x) + L \tag{31}$$

subject to and constraints (25-26)

where $L > 0$ is a large number so that $Z'^{+}(x) - Z'^{-}(x) + L > 0$. The problem (31) is converted to the following optimization problem, by introducing an extra variable x_0 in order to express constraints and objective function as quotient and linear form, respectively.

$$\min\; x_0 \tag{32}$$

subject to $$\frac{Z'^{+}(x) + L}{Z'^{-}(x) + x_0} \leq 1 \tag{33}$$

$$\frac{\kappa^{+}(x)}{\kappa^{-}(x) + 1} = 1 \tag{34}$$

$$x = (S, M, G, T, P) \geq 0 \tag{35}$$

Eqs. (32-35) are transformed to complementary geometric programming (CGP) problems that belong to class of NP-hard nonconvex problems (Chiang et al., 2007). So according to Xu (2014), we introduce an additional variable a and formulated the optimization problem (32-35) as:

$$\min \; x_0 + \omega a \tag{36}$$

$$\text{subject to} \quad \frac{Z'^{+}(x)+L}{Z'^{-}(x)+x_0} \le 1 \tag{37}$$

$$\frac{\kappa^{+}(x)}{\kappa^{-}(x)+1} \le 1 \tag{38}$$

$$\frac{\kappa^{+}(x)}{\kappa^{-}(x)+1} \ge 1-a \tag{39}$$

$$0 \le a \le 1 \tag{40}$$

$$x = (S, M, G, T, P) \ge 0 \tag{41}$$

where ω is the weighting factor with sufficiently large amount. The variable a in problem (36-41) generates negative optimization variables. Therefore, an additional variable b is introduced to convert the variable a into other positive ones:

$$b = \frac{1}{1-a} \ge 1. \tag{42}$$

The following optimization problem is obtained by this transformation strategy:

$$\min \; x_0 + \omega b \tag{43}$$

$$\text{subject to} \quad \frac{Z'^{+}(x)+L}{Z'^{-}(x)+x_0} \le 1$$

$$\frac{\kappa^{+}(x)}{\kappa^{-}(x)+1} \le 1 \tag{45}$$

$$\frac{b^{-1}\left(\kappa^{-}(x)+1\right)}{\kappa^{+}(x)} \le 1 \tag{46}$$

$$b \ge 1 \tag{47}$$

$$x = (S, M, G, T, P) \ge 0 \tag{48}$$

In above problem, the objective function (43) is a Posynomial function, constraints (47-48) are monomial inequalities. They are all permissible equations required in standard GP, while constraints (44-46) are not permitted in a standard GP problem. To cope with this problem, Xu (2014) applied arithmetic–geometric mean approximation in order to approximate each denominator of Eqs. (44-46) by monomial functions. Assume $f(m)$ is a posynomial function as $f(m) = \sum_u v_u(m)$ that $v_u(m)$ are monomial terms. So, we have the following equation with the arithmetic–geometric mean inequality:

$$f(m) \ge \hat{f}(m) = \prod_u \left(\frac{v_u(m)}{\alpha_u(n)} \right)^{\alpha_u(n)}, \tag{49}$$

where n is a fixed point with $n > 0$ and the parameters $\alpha_u(n)$ can be computed as:

$$\alpha_u(n) = \frac{v_u(n)}{f(n)} \qquad \forall u \tag{50}$$

Boyd et al. (2007) showed that $\hat{f}(m)$ is the best local monomial approximation of $f(m)$ near n. Therefore, an inequality restriction on a proportion of two posynomials as $\frac{g(m)}{f(m)}$ can be approximated by $\frac{g(m)}{\hat{f}(m)} \le 1$ while $\frac{g(m)}{f(m)} \le \frac{g(m)}{\hat{f}(m)} \le 1$ holds (Xu, 2014). Using the proposed monomial approximation approach to every denominator of Eqs. (44-46), the following optimization problem is obtained at the i^{th} iteration:

$$\min \ x_0 + \omega b \tag{51}$$

$$\text{subject to} \quad \frac{Z'^{+}(x)+L}{\hat{Z}_0'^{-}(x,x_0)} \le 1 \tag{52}$$

$$\frac{\kappa^{+}(x)}{\hat{\kappa}^{-}(x)} \le 1 \tag{53}$$

$$\frac{b^{-1}\left(\kappa^{-}(x)+1\right)}{\hat{\kappa}^{+}(x)} \le 1 \tag{54}$$

$$x = (S, M, G, T, P) \ge 0 \tag{55}$$

where $\hat{Z}_0'^{-}(x,x_0)$, $\hat{\kappa}^{-}(x)$, and $\hat{\kappa}^{+}(x)$ are the corresponding monomial functions that are calculated by using Eq. (49). Therefore, $\hat{\kappa}^{+}(x)$ has the following formulation and $\hat{Z}_0'^{-}(x,x_0)$, $\hat{\kappa}^{-}(x)$ can be formulated similarity:

$$\hat{\kappa}^{+}(x) = \left(\frac{TN^{-1}}{\upsilon_1}\right)^{\upsilon_1}\left(\frac{\varepsilon T^{2}PN^{-1}}{\upsilon_2}\right)^{\upsilon_2} \tag{56}$$

where $\upsilon_j \ (j=1,2)$ can be calculated by Equation (50) as follows:

$$\upsilon_1 = \frac{T^{(i)}\left(N^{(i)}\right)^{-1}}{T^{(i)}\left(N^{(i)}\right)^{-1} + \varepsilon\left(T^{(i)}\right)^{2}P^{(i)}\left(N^{(i)}\right)^{-1}} \tag{57}$$

$$\upsilon_2 = \frac{\varepsilon\left(T^{(i)}\right)^{2}P^{(i)}\left(N^{(i)}\right)^{-1}}{T^{(i)}\left(N^{(i)}\right)^{-1} + \varepsilon\left(T^{(i)}\right)^{2}P^{(i)}\left(N^{(i)}\right)^{-1}} \tag{58}$$

Now, the problem (51-55) is a standard geometric programming that can be optimized efficiently using GGPLAB solver in MATLAB software (Mutapcic et al., 2006). Also, the proposed algorithm can be summarized as a flowchart in Fig 2.

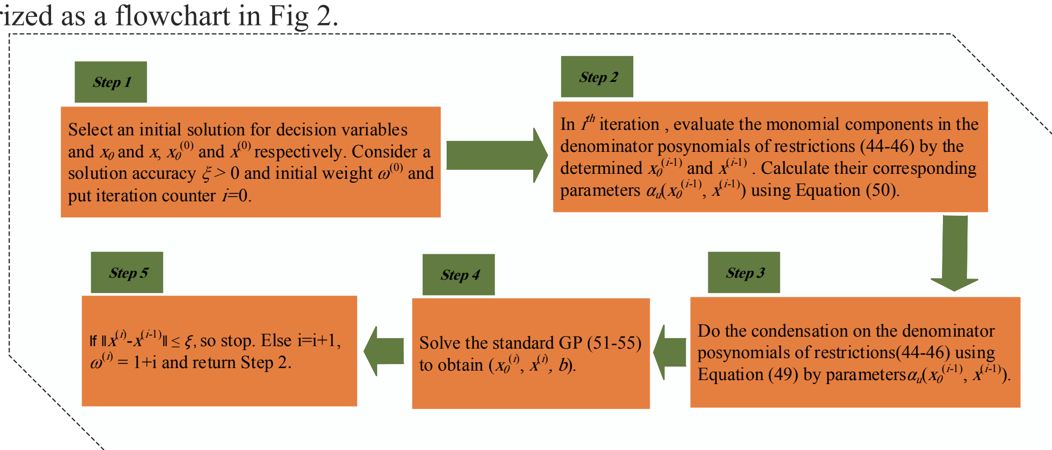

Fig. 2. Flowchart of solution procedure

5. Numerical results

In this Section, an example is designed to demonstrate the application of the model and solution procedure proposed above for a particular retailer that introduces a new commodity to the market and offers a partial delayed payment its customers. The retailer wants to maximize the profit under conditions that demand rate and unit purchasing cost are represented as $\lambda = 1.4\times10^5 M^{0.5} G^{0.01} S^{-3}$ and $P_r = 3Q^{-0.2}$, respectively. The values of parameters for this commodity are given as follows: $A = 200$ (\$/order), $\beta(t) = e^{-t}$, $h = 1.5$ (\$/unit/year), $\pi = 7$ (\$/unit/year), $\pi_l = 3$ (\$/unit), $I_p = 0.05$ (\$/year) and $\eta = 0.2$. The proposed model is solved by using GGPLAB solver (Mutapcic et al., 2006) that is coded in MATLAB R2014b software and implemented on an Intel Core i5 PC with CPU of 1.4 GHz and 4.00 GB RAM. The algorithm parameters are shown in Table 2. The computation results are given in Table 3.

Table 2
The algorithm parameters

L	ξ	Initial solutions					
		$x_0^{(0)}$	$M^{(0)}$	$G^{(0)}$	$S^{(0)}$	$T^{(0)}$	$P^{(0)}$
5.0311	10^{-3}	1540	0.1	0.2	3	1	0.5

Table 3
The computation results

Decision variables								
M^*	G^*	S^*	λ^*	T^*	P^*	Q^*	B^*	Z^*
0.0669	0.0350	7.5530	81.2737	1.4660	0.9812	119.1496	2.2092	285.2565

5.1. Sensitivity analysis

In order to investigate the effect of the changes in some main parameters on the optimal solutions obtained by the global optimization approach, a sensitivity analysis is performed. We first investigate the sensitivity analyses on the optimal solutions due to the parameters h, A, π, η, and I_p. The results of this sensitivity analysis are reported in Table 4 and the following results can be viewed:

- An increasing in the parameter h leads to an increase in M^* and G^*, a decrease in the values of T^*, Q^*, S^*, P^*, and Z^*.
- By increasing of the backordering cost, π, the values of Z^*, Q^*, S^*, P^*, and T^* decrease. Whiles, the amount of M^* increases and the value of G^* is not sensitive to changes in π.
- By increasing of the ordering cost, A, the value of G^*, S^*, P^* and Z^* decrease. Whiles, the values of M^*, T^*, and Q^* increase .
- When η increases, the values of M^*, G^*, S^*, and Z^* increase, whiles the values of T^*, P^*, and Q^* decrease.
- When I_p increases, the values of M^*, G^*, Z^*, T^*, Q^*, and P^* decrease, while the value of S^* increases.

Table 4
Sensitivity analysis on the parameters h , A , π , η , and I_p

Parameter	Value	M^*	G^*	S^*	T^*	P^*	Q^*	Z^*
h	1.125	0.0639	0.0349	7.4692	1.4662	0.9818	120.3852	286.2557
	1.875	0.0676	0.0350	7.5842	1.4558	0.9808	118.4035	271.2758
	2.25	0.0694	0.0351	7.5913	1.4527	0.9801	117.4354	260.7208
π	5.25	0.0657	0.0350	7.5580	1.4677	0.9816	120.0207	319.2078
	8.75	0.0678	0.0350	7.4985	1.4646	0.9808	112.1885	282.8945
	10.5	0.0680	0.0350	7.4611	1.4633	0.9805	106.1469	247.9599
A	150	0.0535	0.0351	7.8960	1.4520	0.9813	92.3403	288.4794
	250	0.0716	0.0349	7.4821	1.4693	0.9810	127.0395	276.2532
	300	0.0740	0.0347	7.1542	1.4722	0.9808	148.0178	275.4421
η	0.15	0.0660	0.0350	7.5512	1.4875	0.9816	120.1489	280.4579
	0.25	0.0680	0.0351	7.5541	1.4602	0.9806	118.3145	289.0143
	0.3	0.0684	0.0353	7.5554	1.4588	0.9799	117.0157	293.2150
I	0.0375	0.0671	0.0355	7.5512	1.4669	0.9816	121.0164	291.2469
	0.0625	0.0663	0.0348	7.5548	1.4643	0.9810	118.8643	281.3648
	0.075	0.0660	0.0343	7.5568	1.4638	0.9808	117.6387	276.0156

We also consider the effect of the changes in values of δ , χ , and α on the total profit . The calculated results are shown in Figs 3-5. We observe from Fig. 3 that when the amount of χ increases the total profit decreases. This is because when χ increases, marketing expenditure increases, so the retail price will be incresed to make up the profit. Incresing in selling price leads to a decrase in demand rate and order quantity, since the total profit decreases. Moreover, when the amount of α increases the total profit decreases (see Fig 4).This is because when the price elasticy to demand increases, demand rate and order quantity decrease; thus, the total profit decreases. In contrast, when the amount of δ increases, the total profit increases and then decreases.

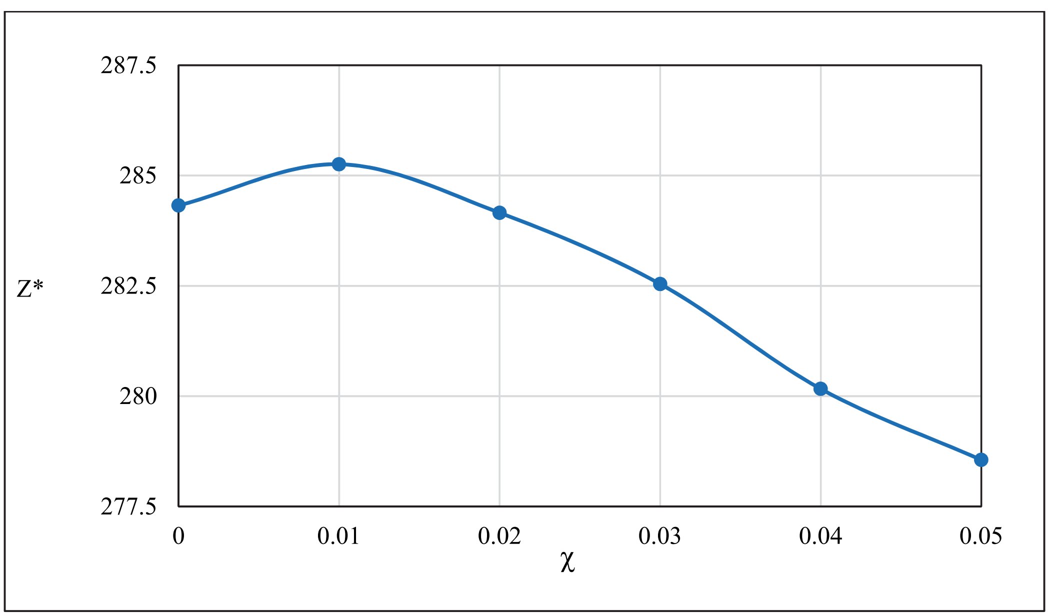

Fig. 3. The effect of change of χ on the total profit

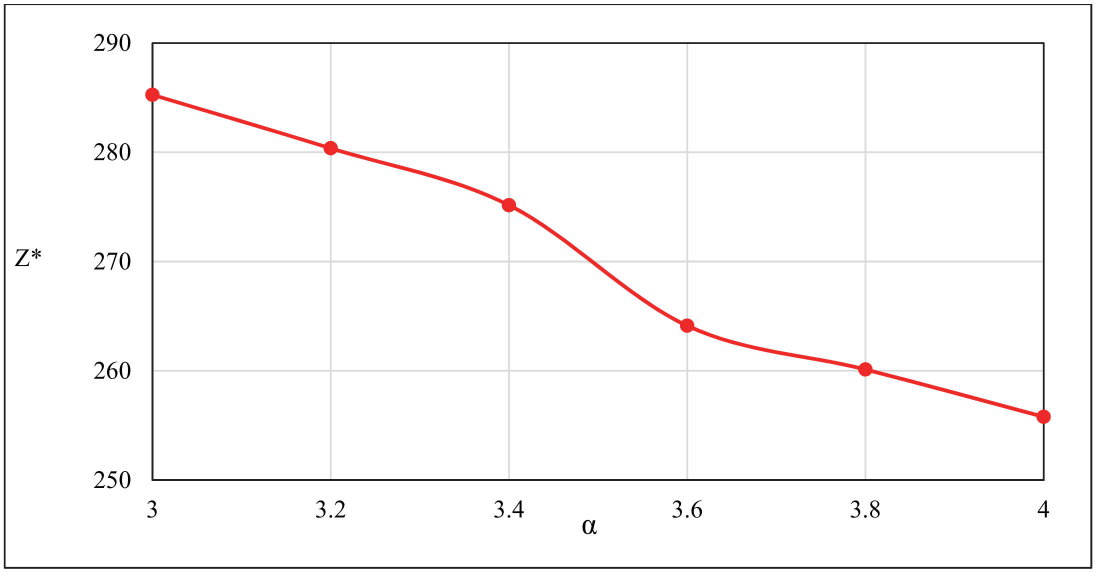

Fig. 4. The effect of change of α on the total profit

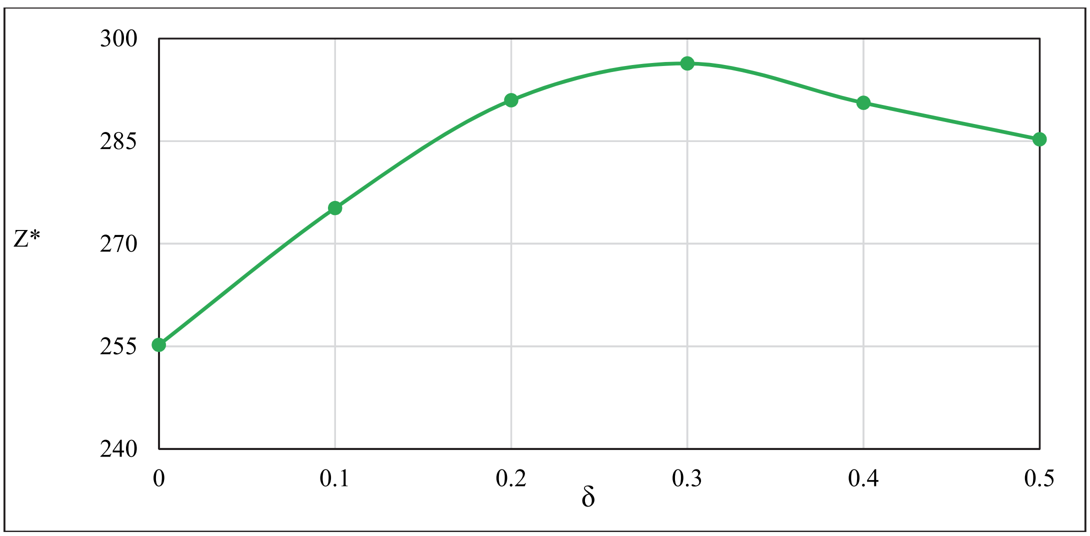

Fig. 5. The effect of change of δ on the total profit

6. Conclusion

In today's business transaction, a permitted delay in payment is offered by buyers that can be considered as a kind of discount and has a positive effect on the demand rate. Hence, we have developed an inventory model in a supply chain by considering shortages and delayed payments in partial form where demand rate was represented as a multivariate function of credit period, marketing expenditure and selling price and also the unit cost was linked to the order quantity. Under these assumptions, the proposed problem has been formulated as SGP problem and we have applied a global optimization approach to obtain global optimal solutions. Finally, numerical examples have been used to demonstrate the proposed model and also sensitivity analysis of important parameters are executed. For future study, the proposed problem can be developed in some ways such as, by considering a fuzzy environment, to allow for inflation, deterioration, quantity discount, to consider the impact of other parameters on unit cost and demand rate.

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Appendix. SGP problems

A SGP program is equal an optimization problem as follows:

$$\min \xi_0(y)=\sum_{k=1}^{n_0}\theta_{0k}c_{0k}\prod_{i=1}^{m}y_i^{\alpha_{oik}}, \qquad c_{0k}>0,\ \theta_{0k}=\pm 1$$

subject to

$$\xi_j(y)=\sum_{k=1}^{n_j}\theta_{jk}c_{jk}\prod_{i=1}^{m}y_i^{\alpha_{jik}}\le 1, \qquad c_{jk}>0,\ \theta_{jk}=\pm 1,\ \alpha_{jik}\in R,\ j=1,2,\ldots,t$$

$$\xi_j(y)=\sum_{k=1}^{n_j}\theta_{jk}c_{jk}\prod_{i=1}^{m}y_i^{\alpha_{jik}}=1, \qquad c_{jk}>0,\ \theta_{jk}=\pm 1,\ \alpha_{jik}\in R,\ j=t+1,t+2,\ldots,o$$

$$y_i>0, \qquad i=1,2,\ldots,m \tag{1}$$